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ACOUSTICS

A Text on Theory and Applications

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PREFACE

THE theory of Acoustics has great beauty. It also possesses a peculiar satisfaction for the student because of the concreteness of the fundamental assumptions and the relative certainty of their essential correctness. Moreover, in recent years, the study of acoustics has become of greater practical importance. In general the theory has not preceded application: speech and musical instruments have been developed by the experimental method and the same is true of the earlier stages of architectural acoustics. But in recent years, with the advent of radio and the loud speaker, of keener interest in the acoustic side of telephonics and of the use of acoustic devices in national defense, development has been guided by theory. This more extensive use of acoustics demands a greater availability of clear exposition of the essential parts of the theory and a presentation with the modern applications uppermost in mind. For this purpose the present work has been prepared. It is an outgrowth of special lectures given in a graduate course in Electrical Communications at Yale University by the first named author, but its final form has been made possible largely through the efforts of the second author, who has at the same time used the manuscript as a text in a graduate course.

The authors have drawn heavily upon classical works for theory (Rayleigh in particular, as must of necessity be the case in any text on theoretical acoustics) and upon the researches of the past decade for applications and additional theory. This indebtedness is amply indicated by references throughout the text and need not be further emphasized here.

A book on the theory of acoustics might well have a thread of mathematical continuity running throughout. But the authors believe that adherence to this policy would defeat the purpose of the present work, for here the applications are made the foci of interest in the theory. On this account the material is presented more nearly by what is known in legal education as the "case method." Nevertheless, the whole is tied together by ample cross references which should assist the reader in the use of the book.

The first chapter contains the discussion of a number of inter-

esting phenomena that are to be carried in mind throughout any consideration of acoustic applications. At the close of this chapter there is introduced the fundamental theory of acoustic waves in fluids. These last-named sections need not be mastered by the student at his first reading, but they contain deductions of the basic formulas which are used throughout the remainder of the text. Some of the more complicated theoretical deductions are relegated to the Appendices in order to prevent undue burdening of the text, while certain aspects of acoustical theory are omitted entirely since they are readily available in numerous treatises. In general, the endeavor has been to stress physical aspects and principles rather than formal mathematical procedure. Nevertheless, no effort has been spared to make the analysis, wherever given, clear and convincing.

The present work is unique in its stress on the important researches of the past decade and in the useful combination of material both from the theoretical and practical viewpoints. Attention may be called to the emphasis placed on such topics as the general problem of acoustic transmission through conduits, the effects of branch lines, acoustic wave filters, acoustic coupling, submarine signalling, and physiological and atmospheric acoustics. In the discussion of these and other topics every effort has been made to bring the book up to date, so that the student and general reader will obtain a broad as well as a detailed picture of the present activities of the science of acoustics.

The authors desire to acknowledge their indebtedness to the stimulating interest of Professor H. M. Turner of Yale University, at whose invitation the material for the course of special lectures for his students was first prepared. The second author wishes particularly to express his deep appreciation of the great encouragement and assistance given by his colleagues of the Department of Physics at Yale University during the preparation of the manuscript, which was completed while he was a member of that department. He also desires to express his gratitude to his wife, Rachel T. Lindsay, for very material assistance in the preparation of the manuscript and the reading of the proof.

G. W. S.
R. B. L.

July, 1930.

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CHAPTER I

Some Simple Properties of Acoustic Waves—Fundamental Theory

1·1. Definition of Acoustic Wave.—The name " wave " is given to any propagated alteration in the physical condition of a medium. In acoustics it may be thought of as a propagated change in amplitude of *pressure*, *particle displacement*, *particle velocity or of density*. Thus, if the particle displacement be denoted by ξ, the most general expression for the propagation with velocity c of a plane wave of particle displacement in the positive x direction is

$$\xi = f(ct - x),$$

where f is *any arbitrary* function of the argument $(ct - x)$. At the point $x = x_0$ at time $t = 0$, the displacement is $f(-x_0)$, while at the later time $t = t_0$, the displacement is $f[ct_0 - (ct_0 + x_0)] = f(-x_0)$ at the point $x = ct_0 + x_0$. In other words the disturbance at x_0 has traveled to $x_0 + ct_0$ in the time t_0, corresponding to progression in the positive x direction with velocity c. In acoustics we have for the most part to deal with *simple harmonic* plane and spherical waves, i.e., wherein f is a circular function. For example, it will be shown in Sections 1·12, 1·13, 1·14 (see Eqs. (1·23) and (1·42)) that for a simple harmonic plane wave travelling in the x-direction the *excess* pressure (i.e., the difference between the actual pressure at a point at any instant and its equilibrium value) is given by

$$\delta p = kc\rho_0 A \sin k(ct - x),$$

while for a simple harmonic spherical wave,

$$\delta p = \frac{kc\rho_0 A'}{r} \sin k(ct - r),$$

wherein the various quantities are defined as follows:

t, the time elapsed,
x, the distance from the fixed origin of rectangular coordinates,
r, the distance from the sound source,

A, A', constant amplitude factors, i.e., amplitude of the velocity potential, the quantity whose gradient is the particle velocity. (See Sec. 1·12.) A thus has the dimensions of $[L]^2/[T]$. A'/r has the same dimensions in the case of spherical waves,

ρ_0, mean equilibrium density of the medium,

c, the velocity of the wave,

k, constant $= \frac{2\pi}{\lambda} = \frac{\omega}{c}$,

where λ = wave length or distance between two successive maxima, and $\omega = 2\pi\nu = 2\pi$ times the frequency of vibration.

The above are the two types of waves which are most important in acoustics. Of the two, more use of plane waves is made in practice. For spherical waves at a great distance from the source may approximately be treated as plane, and the same is true of waves passing through a tube whose cross section is not rapidly varying. Cylindrical waves are treated in some texts [1] but we shall not emphasize them here. An ample discussion of spherical waves will be found important, however, in the case of sound signalling (Chap. X).

1·2. Reflection from a Plane Wall.—Acoustic Image.—The actual vibrations in acoustics are too complicated to follow in one's imagination. The fundamental equations to be deduced in the latter part of this chapter relieve one of any need of attempting to envisage the details. But even with the general conception of wave motion one can scarcely picture the effects of superposing a multitude of waves. Consider for example the difficulty involved in computing for any point the effect of a source placed in front of a rigid plane wall of infinite extent. The resultant effect is caused by the waves coming directly from the source and by the waves reflected from every part of the area of the wall. Such a computation would be unnecessarily involved, for by the brief consideration that follows it is possible to simplify greatly the theory. This resort to general reasoning in advance of mathematical processes is usually very fruitful, as this example beautifully illustrates.

Consider a source of sound O near a plane wall AB (Fig. 1·1). The question arises, what is the effect of the wall on the sound waves? Draw the line through O normal to the wall and consider

[1] See, for example, Lamb, Hydrodynamics, 1916, p. 520.

the point O' on this line as far behind the wall as O is in front of it. Then imagine a like source of sound placed at O' of the same frequency and phase [1] as that at O, and suppose the wall removed. Now at any point P on the imaginary plane normal to the line OO' and passing through its midpoint, the resultant displacement due to the two sources O and O' will at any time be in the plane, for the components normal thereto cancel each other. Hence, we can now replace the plane by a hypothetical rigid wall of zero thickness without altering the resulting wave motion, showing that, on the right hand side, the effect of the wall on the sound from O is the same as would be produced by the introduction of a like source at O' and the removal of the wall. The reflection of a spherical wave from an infinite wall would, at first glance, seem like a complicated problem, for the effect of the reflection from every point of the wall must be considered. But the above reasoning simplifies the whole matter to the computation of the effect of two spherical waves. We shall call O' the acoustic image of O, by which we may determine quantitatively the reflection of sound from the wall. The reflected sound, that is, the sound from the image, is called the echo.

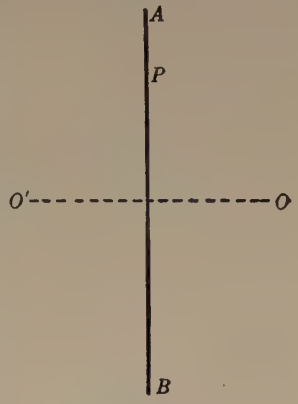

FIG. 1·1.

The practical importance of natural echoes has recently been realized in connection with the detection of icebergs by the reflection of underwater sound waves from their surfaces, and the measurement of the depth of the sea by the echo therefrom. Both these will be discussed further in Chapter X. Interesting material on natural echoes will be found in the work of Tyndall.[2]

It is clear that, in the discussion of reflection, the wall above considered need not be geometrically plane in order that an image may exist. It may be rough provided that the variations from

[1] Two sources of sound of the same frequency are said to be in the same phase when the maximum disturbance at each source takes place at the same time.

[2] See, for example, Tyndall, Lectures on Sound, 1867, p. 17 ff. See also the description of aerial echoes in the same author's article on fog signals in "Fragments of Science."

planarity are small compared with a wave length of the reflected sound. Naturally, the "rougher" the surface the more diffused will be the image.

A multitude of echoes following closely upon one another produce the effect known as reverberation. This is apt to be of special moment in large closed spaces where the reflecting surfaces are widely distributed, and constitutes a very important problem in auditorium design and architectural acoustics generally. Details on this point will be found in Chapter XI.

1·3. Huyghens' Principle and Application to Practical Reflectors.—To discuss reflection in a more general fashion we need the concept of wave front. The wave front is a surface so drawn that at all its points the wave has the same phase;[1] that is, the particles of the medium on this surface reach their maximum positive or negative displacements at the same instant. Thus, for example, the wave fronts of the sound wave from a point source in a homogeneous medium are spheres with the point as center. If in any particular case the wave front is given at a certain instant of time, that for any subsequent instant can be found by means of an important theorem known as Huyghens' principle. This states in effect that each point on a wave front may be assumed to be the source of a hemispherical wavelet and that the new wave front is the mathematical envelope of these wavelets at any subsequent instant. This principle can be stated in more exact form[2] and rigorously derived, but we shall be content here with the above statement.

From optics we know that when light is allowed to pass through a hole in a wall we get a more or less sharply defined "beam." In acoustics also it is possible to produce a beam of sound if, as in optics, the dimensions of the hole are large compared with wave length. For example, consider a plane wall with a vibrating area of dimensions satisfying the above condition (Fig. 1·2). For points along $A'B'$ (i.e., in a beam normal to the wall) the Huyghens' wavelets will reinforce each other, while for points outside this beam, such as P, there will be destructive interference with consequent loss in intensity. Most of the resulting sound may therefore

[1] A careful discussion of the meaning of the expression "phase of a wave" will be found in Section 1·17.

[2] See, for example, Rayleigh, Theory of Sound, II, 1926, § 283. Also Drude, Theory of Optics, Eng. Ed., 1902, p. 179 ff.

be said to be confined within a region bounded (roughly) by AC and BD. If the wave length is decreased a sharper beam results; increasing the wave length causes the beam to spread out and the resulting wave front becomes more and more nearly hemispherical.

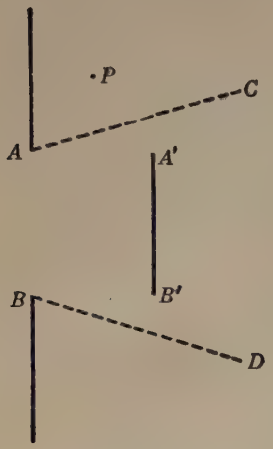

FIG. 1·2.

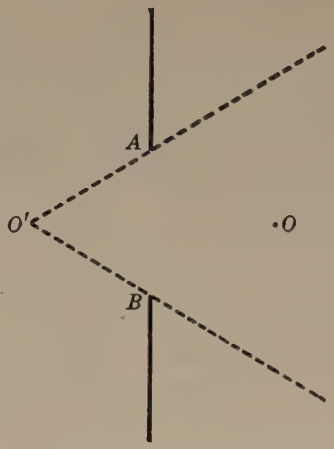

FIG. 1·3.

The above discussion leads at once to a consideration of the reflection of sound from finite areas. Consulting for this purpose Fig. 1·3, we see from the preceding paragraph that the reflection from a plane area AB (here assumed circular for convenience) of the sound from a source O is equivalent to that part of the sound from the image O' which is included within the cone $AO'B$. (Note that the total sound from O' is strictly equivalent to reflection from an infinite wall.) Now if the diameter AB is large compared with the wave length, the reflected sound will remain approximately within the cone and we shall have a reflected conical beam. But if the diameter AB is much less than a wave length there will be little phase difference in any direction such as P, Fig. 1·2, of waves from different parts of the reflector, and consequently little interference in any direction at all. This means that the reflection will not be in the form of a beam, but that the sound will be scattered in all directions. Between these two extremes there are gradations from a well defined beam to diffuse scattering. It is worth noting that decreasing the size of a reflector lowers its effectiveness in two ways, viz., (1) in diminishing the actual amount of sound reflected,

and (2) in scattering in all directions that which is reflected. Conversely we may make the important statement that the effectiveness of a small reflector increases much more rapidly than its area.

Coming now to practical mirrors other than plane, it is a well known optical fact that, for a parabolic mirror, a wave front normal to the axis is brought to a focus at the mathematical focus of the mirror. More careful study shows that in the actual case this is not a mathematical point but merely the region of phase agreement and consequent reinforcement of the reflected waves. This region must therefore have dimensions of the order of a single wave length. From the relatively long length of sound waves, it is seen that sound focusing in such a mirror is very rough compared with the case of light. Thus, for a frequency of 100 cycles the focal spot, as it is called, is of the order of 1090/100 ft. = 11 ft. approximately, while even for the rather high audible frequency of 1000 cycles it is 1.1 ft. approximately. Similar remarks pertain to elliptical reflectors. Both types show a definite selectivity with regard to frequency, focusing high tones much more sharply than low ones.

As a fact which has an important bearing on sound signalling in water, it should be noted that, because of the greater velocity of sound in water, the wave length of a sound of given frequency is about four and a half times as great in water as in air. From the above discussion it is clear that a given reflecting area under water scatters sound of given frequency to a greater degree than the same area in air. Conversely, for good reflection in water one must use larger reflecting surfaces than are necessary in air.

This is an appropriate place to point out that whereas, according to the common view, horns and trumpets owe their properties to the collection of sound by reflection from their inner surfaces, the above discussion shows that such optical-like reflection does not occur. The real explanation of the effectiveness of these instruments (i.e., their resonating properties) is reserved for a later section. (See Chap. VI.)

1·4. Selective Property of Plane Reflectors.—The use of Fresnel's zones in optics is well known. Their application to acoustic waves leads to the consideration of an interesting selective property of reflectors. Consider the circular disc *ACB* in Fig. 1·4 and suppose that all points of it are vibrating in the same phase. The total effect at the pole *O* will be that due to the superposition of the contributions from each point of the disc. Draw about *C* as a

center a circle of radius *CD* such that the difference between *OD* and *OC* is small compared with one wave length of the sound being produced by the disc. Then, draw other circles about *C* in such

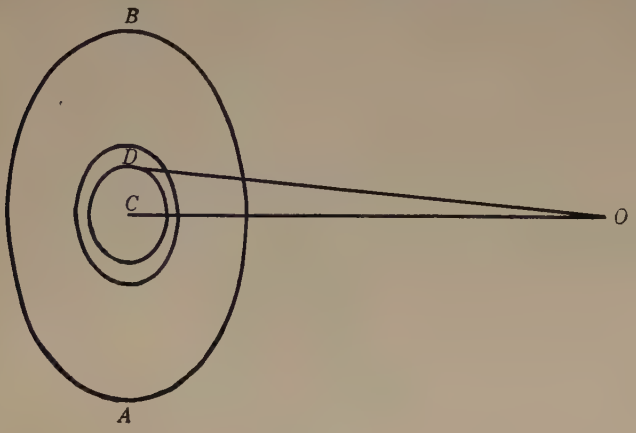

Fig. 1·4.

a way that each annulus has the same area as the central circle. This will obviously be the case if the radii are chosen in the ratio of the square roots of the natural numbers. Each annulus may be thought of (for the moment at least) as contributing approximately the same amplitude of vibration at *O*, but with differing phase. Hence the contributions must be added vectorially and the result of the summation will be to increase the effective amplitude at *O* until we reach an annulus such that the difference between its distance from *O* and the distance *CO* is exactly one half wave length. The following figure (Fig. 1·4*a*) will help to make this clear. If we represent by *Oa* the amplitude of vibration at *O* produced by the inner circle, then *ab* will represent that produced by the first annulus, and the sum of the two vectors *Oa* and *ab* (namely

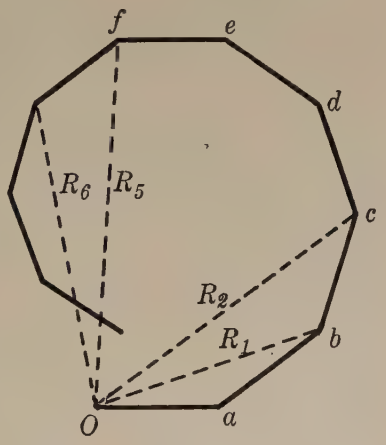

Fig. 1·4*a*.

R_1) will represent the resultant effect at O due to both circle and annulus. Similarly as further annuli are considered, the vector summation tends to increase the magnitude of the resultant until we reach that annulus which produces an amplitude at O exactly out of phase with Oa. This is represented in the figure by ef. The resultant having reached a maximum now tends to decrease as further annuli are considered, and so the effective amplitude at O, as is clear from the figure, will become a minimum when we reach that annulus for which the difference between CO and its distance from O is one whole wave length. This minimum is not exactly zero, since as the points on the disc get farther from O their influence does decrease slightly (with the inverse square of the distance, as a matter of fact). This discussion then indicates that as the area of the disc increases, the effect at O passes through a series of maxima and minima which depend on the frequency of the sound. The same considerations apply, of course, to a plane wave reflected normally from a plane mirror, wherein the effect at O is thought of as due to the sound reflected from the disc ACB. It thus develops that, for a given frequency of sound, there is an optimum size of mirror to give the greatest effect at O. Conversely, a mirror of definite size will produce maximum effect at O for one definite frequency.

As a practical example of this selective property the pinnae or auricles of the ear suggest themselves. Due to their relatively small size, however, it is clear that for sounds of ordinary wave length and pitch their effect is small. For a sound of a frequency of 10,000 cycles, on the other hand, their influence is very noticeable, as may be observed by holding a watch (the ticks of which contain some high frequency components) first in front of the head and then behind the head. In the same connection we should note the effect of "cupping" the ears with the hands.

The use of reflectors is not limited to points distant from the source, but frequently occurs at the source of the sound itself. For example, it will be shown in Section 1·19 that if a source is placed at an infinite wall, the output is greatly increased. From the discussion in this section, it is observed that even if the reflector is small, there would be a gain in output. Such a reflector is called a "baffle plate" and is found in use in the flat rim surrounding loud speakers, both of the horn and cone type. The effect of such a plate at a distant point O can be determined by a computation

following the suggestions above. Consider a source at the point C in Fig. 1·4 and the effect at O of reflection in the neighborhood of A. The path of the sound must be considered to be from C to A and A to O. The phase difference produced by this length of path (and the reflection) is then utilized in computing the effect at O. Otherwise, the method follows that discussed above. It is obvious that baffle plates not only reflect but also form shadows and have a screening effect.

1·5. Reflection at a Change in Area of a Conduit.—While we are considering the general problem of sound reflection, it will be worth while to note that a sudden change in the cross sectional area of a pipe through which sound is being transmitted will give rise to a reflected wave. Thus consider the diagram (Fig. 1·5) with

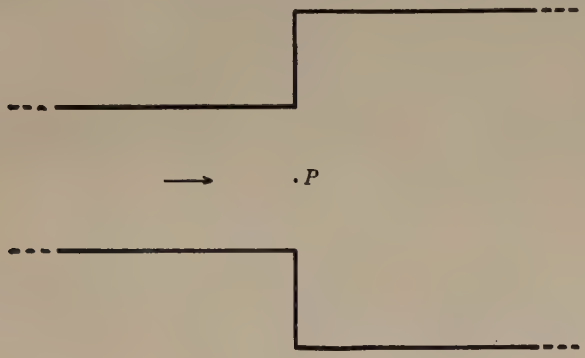

FIG. 1·5.

sound traveling in the direction of the arrow. The cross sections are S_1 and S_2 respectively. At the point P in the junction, the following boundary conditions must be fulfilled: (1) the pressure at P must be the same on both sides, i.e., there must be no discontinuity in pressure as we pass through P, and (2) the rate of volume displacement of fluid at P is the same on both sides (the ordinary continuity principle in hydrodynamics). In a later section (Sec. 3·5) it will be rigorously proved that the application of these conditions leads to the following expression for the ratio of the reflected intensity to the incident intensity:

$$\left(\frac{S_2 - S_1}{S_2 + S_1}\right)^2.$$

This value, being independent of the sign of $S_2 - S_1$, is the same for either direction of the wave.

1·6. Velocity of Sound.—It will be shown in Section 1·13 that the general expression for the velocity of a compressional wave in a compressible fluid is

$$c = \sqrt{\frac{dp}{d\rho}},$$

which for the case of adiabatic change becomes

$$c = \sqrt{\frac{\gamma p}{\rho}},$$

where γ is the ratio of the specific heat at constant pressure to that at constant volume. The general gas equation is

$$pV = RT, \tag{1·1}$$

where V denotes volume, T the absolute temperature, and R is the gas constant. Since ρ varies inversely as V, it is seen that if T remains constant p/ρ is constant and the velocity of sound in a gas is independent of change in pressure. However, the velocity does depend on temperature. For, introducing the mass m, we have the general expression

$$c = \sqrt{\frac{\gamma RT}{m}}. \tag{1·2}$$

Therefore, if we denote by c_0 the velocity of sound in air at 0° C and by c_t the velocity at t° C, and apply eq. (1·2), we have

$$c_t = c_0 \sqrt{1 + \frac{t}{273}}. \tag{1·3}$$

The velocity of sound in a liquid like water also depends on the temperature. The *general* expression (Sec. 1·13, eq. (1·30)) for the velocity is

$$c = \sqrt{\frac{E}{\rho}},$$

where E is the volume elasticity. Using the compressibility $K = 1/E$ instead, we have

$$c = \frac{1}{\sqrt{K\rho}}. \tag{1·4}$$

Now both K and ρ (for temperatures above 4° C) decrease with increasing temperature, though the relations are, of course, em-

pirical. Thus for low pressures in sea water we have:[1]

$$K = 481 \times 10^{-7} - 340 \times 10^{-9}t + 3 \times 10^{-9}t^2,$$

where t is the temperature on the centigrade scale. There is no empirical formula for the dependence of ρ on the temperature for a very wide range of temperature. A few selected values for distilled water will suffice. Thus if the density at 4° C is taken as 1.00,[2] that at 15° C is .99913, at 30° C is .99567, at 60° C .98324, and at 100° C it is .95838.

1·7. Convective Refraction.—The refraction of sound waves in a single medium like the atmosphere can take place in two ways, viz., by the effect of wind and by the effect of temperature variations from place to place. The former phenomenon is known as convective refraction. We shall discuss it first. Consulting Fig. 1·6, let AB denote the boundary between two adjacent regions of

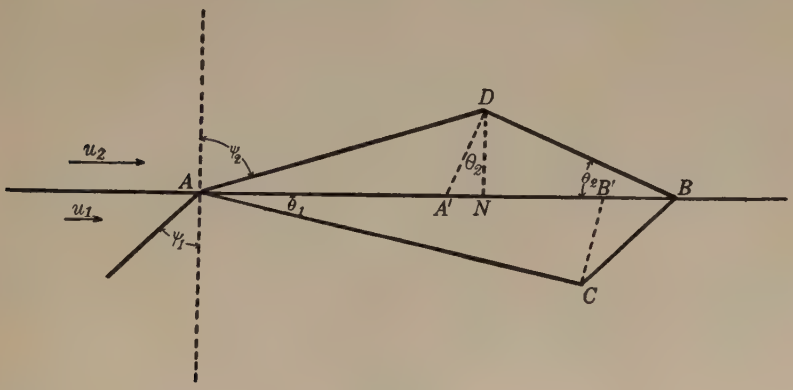

FIG. 1·6.

air in the lower of which the wind velocity is u_1 (supposed to be in the direction AB) and in the upper of which it is u_2, where $u_2 > u_1$; these velocities are relative to the stationary axes in Fig. 1·6. Let a plane wave front AC meet the boundary at A, with angle of incidence θ_1. If there were no wind the direction of propagation in the lower region would be simply CB' normal to AC. The effect of the wind is to carry the medium bodily and hence to shift this direction to CB, where $BB'/CB' = u_1/c$, c being the velocity

[1] See Aigner, Unterwasserschalltechnik, 1922, p. 46.

[2] Smithsonian Tables.

of sound in air. If the time from C to B be τ, then $\tau = CB'/c = BB'/u_1$. To obtain the wave front in the upper region we note that in time τ the upper medium moves $AA' = u_2\tau$. The position of the wave is hence the same as if it had originated at A' and in time τ the disturbance originally at A is at D, where $A'D = c\tau$; DB can be shown by taking intermediate points to be the wave front in the upper region and AD is the direction of propagation. It is interesting to note that in neither region is the direction of propagation normal to the wave front, a fact due to the motion of the medium with respect to the observer.

From Fig. 1·6, the relation between θ_1 and θ_2, i.e., the law of refraction, may be readily obtained. For we have simply:

$$\frac{A'B}{A'D} = \frac{AB' + B'B - AA'}{c\tau} = \frac{AB'}{c\tau} + \frac{u_1 - u_2}{c},$$

that is,[1]

$$csc\ \theta_2 - csc\ \theta_1 = \frac{u_1 - u_2}{c}. \tag{1·5}$$

If the "ray" directions are given by ψ_1 and ψ_2 respectively we can also ascertain their values in terms of θ_1 and θ_2. For if we draw DN normal to AB, $\angle A'DN = \theta_2$, and it follows that

$$\tan \psi_2 = \tan \theta_2 + \frac{u_2 \sec \theta_2}{c}. \tag{1·6}$$

There is an exactly similar equation, with all the subscripts unity, for the incident ray.

If the values are such that

$$csc\ \theta_1 + \frac{u_1 - u_2}{c} < 1,$$

then, since there is no value of $csc\ \theta_2$ that is less than unity, the reflection must be total. The critical angle is, of course, that for which $csc\ \theta_2 = 1$.

It should be emphasized that the above discussion is by no means complete, since in the actual case there is no sharp boundary between air regions of differing wind velocity. The variation is always a more or less gradual one. Nevertheless a more detailed study can follow the above lines by dividing the medium into thin strata.

[1] We shall derive a more general equation for sound refraction in Chapter XII (Atmospheric Acoustics). (See Sec. 12·2.)

Keeping in mind the discussion in this section the reader will have no difficulty in understanding why sound in air will often pass more readily in one direction between two points than in the other, and also why elevated sources are very advantageous in transmitting to windward. It must be borne in mind, however, that in any discussions such as the above we do not have a beam of sound as in light. For example, one could not secure a plane sound wave sharply limited to a given area of cross section, and consequently could not secure total reflection. As has been noted before and as will be emphasized again, the production of a beam of sound, due to its long wave length, is in general a difficult matter.

1·8. Temperature Refraction.—Let two strata of air with boundary AB' (Fig. 1·7) be assumed at rest and at temperatures t_1 and t_2 respectively, with corresponding sound velocities c_1 and c_2, where $c_2 > c_1$. The incident wave front is AB in the stratum of temperature t_1 and the angle of incidence is θ_1. The refracted wave front is $A'B'$ with angle of refraction θ_2. The construction of the refracted wave front follows at once on the application of Huyghens' principle, for $AA'/BB' = c_2/c_1$ and in this case the rays are normal to the wave fronts. The law of refraction is

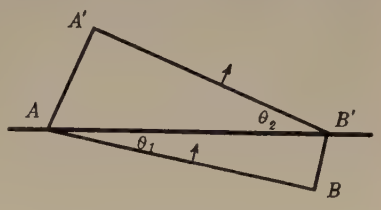

FIG. 1·7.

$$\sin \theta_2 / \sin \theta_1 = c_2/c_1, \qquad (1·7)$$

which is the ordinary law of Snell in optics. Naturally, in practice there is no sharp boundary but a more or less continuous variation in temperature with a corresponding continuous bending of the wave front. Usually the effects of both wind and temperature occur simultaneously.

From the discussion above it is seen that a negative temperature gradient (i.e., coldest air nearest the ground), such as obtains near the surface of the earth in the early morning hours after a clear night, tends to produce a bending of sound wave fronts toward the earth, increasing the range of sound transmission. The ordinary day time condition is a positive gradient and in this case the wave fronts are bent away from the earth, considerably reducing the range. Here again wind introduces a complicating factor.

1·9. Scattering by Selective Reflection and Refraction.—In view of the discussion of the preceding two sections it is not surprising that atmospheric conditions exercise an important influence on the propagation of sound in the air. The various strata of the atmosphere are not always horizontal or even plane. In the case of prominent irregularities, we should then expect a great deal of scattering by reflection and refraction. Moreover, this scattering should be selective, i.e., greater for short wave lengths than for long wave lengths. This has actually been found to be true in experiments [1] on airplane detection under what may be termed "poor listening" conditions. In these tests the sound from an airplane at the greatest hearing distance was found to be limited to the lowest frequencies in the emitted complex sound. The effect of the scattering on the decay of sound intensity was also well illustrated by the same experiments. As will be shown later, the falling off in intensity of a sound wave in a *homogeneous* fluid is proportional to the inverse square of the distance from the source. The listening apparatus used in the experiments just mentioned had an amplification factor of 100, so that under the best conditions one should have heard a sound 10 times the distance it could be heard with the ear alone. As a matter of fact, on bright sunny days with cumulus clouds forming, the airplane noise range was only twice that of the unaided ear. Even under the best night conditions the range was only three times that of the ear alone. Selective scattering also plays an important role in submarine transmission as will be noted later (Chap. X).

The interesting silence areas observed during the propagation of an explosive wave may also be traced to meteorological conditions of wind and temperature. Reference on this point may be made to the interesting theory of Esclangon.[2] Wiechert [3] has developed a theory to account for the same phenomenon by postulating a reflecting layer some 50 km. above the earth's surface and the interference of the normal direct wave and the reflected wave (as witness the analogous phenomenon in radio wave transmission with the

[1] See G. W. Stewart, Phys. Rev., **14**, 376 (1919).

[2] E. Esclangon, Comptes Rendus, **178**, 1892, 1924. For further observations on silence zones, see the work of Dufour, Deslandres, Villard, Maurian, Čollignon in volumes 178 and 179 of the Comptes Rendus.

[3] E. Wiechert, Meteorolog. Zeitschrift, **43**, 81, 1926. For further German work in this field see J. Kolzer, Meteorolog. Zeitschrift, **42**, 457, 1925 and W. J. Witkiewitsch, Meteorolog. Zeitschrift, **43**, 91, 1926.

postulation of the Heaviside layer). Esclangon (loc. cit.) casts grave doubt, however, on this explanation.

More recent investigations [1] seem to indicate that the most probable single explanation for the abnormal zone of audibility is to be found in the reversal of the temperature gradient, producing at heights above 30 km. temperatures which are of the same order as that at the surface. It is evident that such a state of affairs would be effective in producing the necessary bending of the wave fronts to account for the distant audible zone. The presence of such a high temperature layer has already been indicated by the study of meteors and it is believed that the temperature is maintained by the absorption of solar energy by the ozone layer whose center of gravity is in the neighborhood of 45 to 50 km.

1·10. Diffraction.—Sound Shadows.—Any change in the direction of propagation of sound waves, not caused by a variation in the properties of the medium but by the bending of the waves about

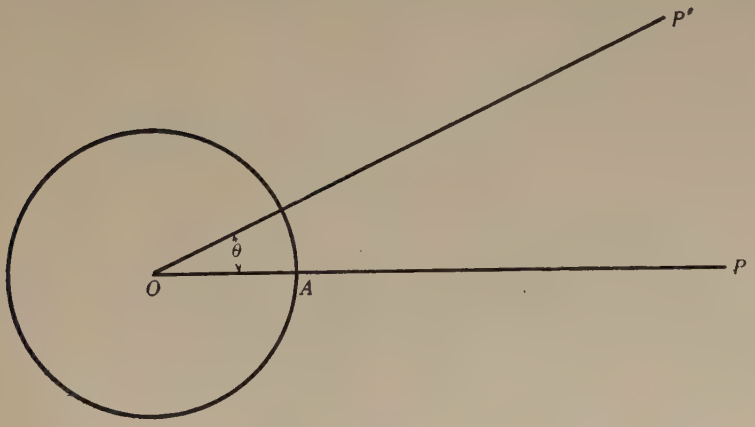

FIG. 1·8.

obstacles, is ascribed to *diffraction*, which is thus an extremely important acoustic phenomenon. On its occurrence depends in large measure our ability to hear sounds from all directions, and there is hardly a technical application of acoustics which does not in some way involve diffraction. One of the most interesting and typical diffraction problems is that of the sound shadow cast by the human head, whether of speaker or auditor. This is most

[1] See, for example, E. H. Gowan, Nature, **124**, 452, 1929 (Sept. 21).

easily treated by the discussion of the ideal case of a rigid sphere.

Consulting Fig. 1·8, let the circle represent the cross section of the rigid sphere. (The word rigid is to be interpreted to mean that the sphere does not vibrate under the influence of any sound which may happen to fall on it.) A is a point source on the sphere, and the problem is to determine the relative intensity of sound at points which, like P and P', are equidistant from the sphere but

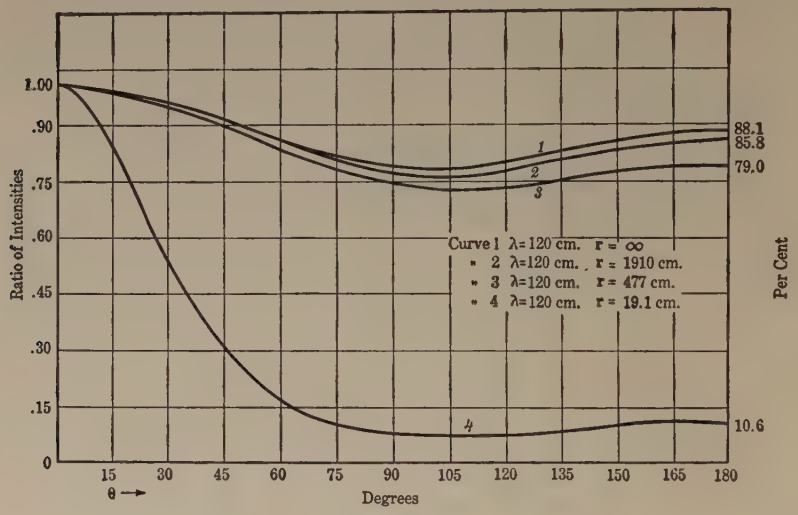

Fig. 1·9.

at different azimuths with respect to the source. The complete theory has been presented by Stewart,[1] following the initial work of Rayleigh and others, and reference should be made to his article for details. Briefly stated, the method of calculation consists in setting up the general equation of wave motion (eq. (1·16) of Sec. 1·12) using the assumption that φ, the velocity potential, is a harmonic function of the time and employing three dimensional polar coordinates. This equation is then solved for φ with the insertion of the boundary conditions imposed by the presence of the sphere. As soon as the value of φ at any point in the neighborhood of the sphere is known as a function of the distance from the sphere and the azimuth, the intensity can be found by a single transformation (for as will be seen later (Sec. 1·15) the intensity

[1] G. W. Stewart, Phys. Rev., 33, 467, 1911. See also R. V. L. Hartley and T. C. Fry, Bell Sys. Tech. Jl., 1, 33, 1922.

is proportional to the square of the condensation and hence to $(\partial\varphi/\partial t)^2$). The actual computations are somewhat complicated, but the results may be clearly seen from an examination of Figs. 1·9 and 1·10.

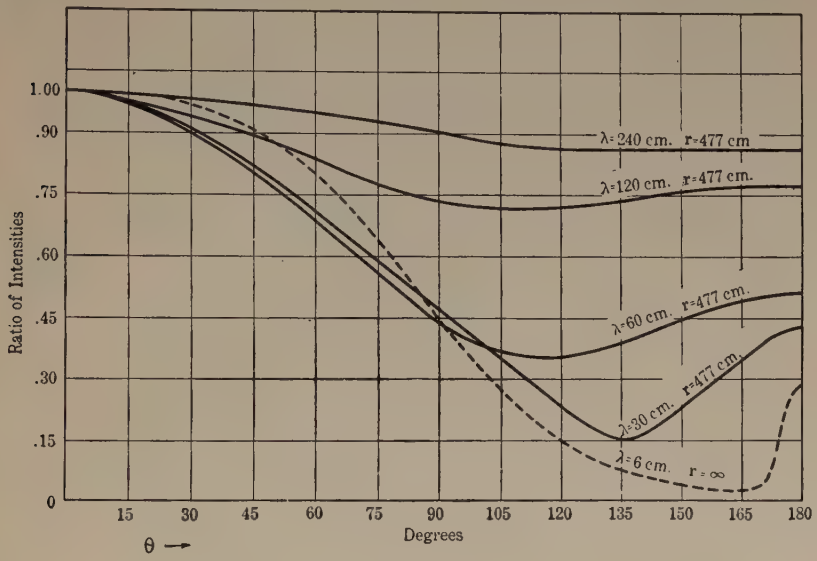

FIG. 1·10.

Fig. 1·9 refers to a sphere whose circular section is 60 cm in circumference and plots for a wave length of 120 cm, the ratio of the intensity at a distance OP' (Fig. 1·8) for any angle θ (from 0° directly in front to 180° directly behind A) to the intensity when $\theta = 0°$. There are four curves, corresponding to four values of r (the distance OP') ranging from $r = \infty$ to $r = 19.1$ cm. The results are rather striking, showing that for great distances the variation in intensity with azimuth is small, but that close to the sphere there is a marked variation, the intensity at 90° falling to less than $^1/_{10}$ that for $\theta = 0°$. In each case, however, there is a maximum intensity directly in front and directly behind.

Fig. 1·10 shows the effect of modifying the frequency while the distance is kept constant. The curves refer to the same sphere as Fig. 1·9, but all the curves (except the bottom one) are computed for a distance of 477 cm from the sphere, the wave lengths ranging from $\lambda = 240$ cm to $\lambda = 30$ cm. As one would expect from the analogous optical case the falling off in intensity with θ is much

more marked for short wave lengths than for long. Here again we notice maxima at 0° and 180°. It is also of interest to note the progressive angular shift in the minimum with decrease in wavelength.

The question next arises: can the above theory give one a clear idea of the shadow cast by the head when the source of sound is at a distance from the head? This problem is fundamental in hearing and in the perception of sound direction. That it can be solved in terms of the discussion in this section is made apparent by the reciprocal theorem of Helmholtz, as is indicated in the next section.

1·11. **Helmholtz's Reciprocal Theorem with Application to Sound Shadow Cast by the Head.—Scattering.**—This theorem may be stated as follows:[1] "If in a space filled with air which is partly bounded by finitely extended fixed bodies and is partly unbounded, sound waves be excited at any point A, the resulting velocity potential φ at a second point B is the same both in magnitude and phase as it would have been at A had B been the source of sound." This theorem is a special case of a very much more general principle of reciprocity elucidated by Rayleigh.[2] It is shown that in its general form the principle holds true when there are dissipative forces present in the medium, i.e., when absorbing surfaces are present, provided only that these forces are proportional to the first power of the particle velocity. Moreover, the fluid need not be homogeneous. On the other hand, in using the restricted principle, one must be careful to note that the sources to which it is applicable are simple and constant sources, i.e., sources which in the absence of an obstacle would generate symmetrical waves.

The reciprocal theorem can be applied to the case of the preceding paragraph by assuming that the source is at the point P' distant r from the head and that the ear is at A. We wish then to ascertain the effect at the ear if the source is moved about the head. From what has been said it is evident that we can apply the reciprocal theorem to intensity values as well as velocity potentials. Hence to solve the above problem we have only to take the intensity values directly from the Figures 1·9 and 1·10, assuming that the head is approximately a rigid sphere with the ears diametrically opposite. The results are plotted in Fig. 1·11, where it should be noted that the ordinates are the relative values of the sum of the

[1] Rayleigh, Theory of Sound, II, § 294, 1896.

[2] Rayleigh, Theory of Sound, I, § 107 ff., 1926.

relative intensities at the two ears as the line joining the ears is rotated through an angle of 180° with respect to the direction of the source of sound. The numbers on the curves refer to the distance of the source and the wave length considered. The most

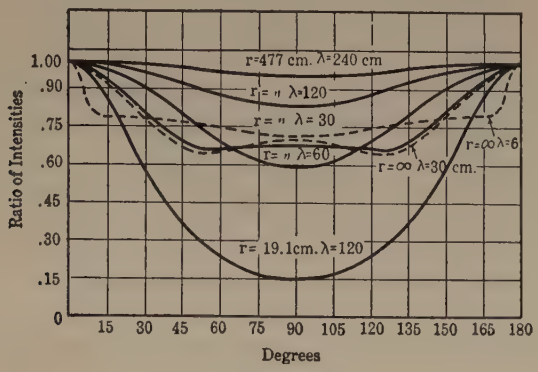

FIG. 1·11.

interesting result is the fact that the resultant apparent intensity is a maximum when the diameter connecting the ears is in the direction of the source of sound, with the minimum occurring at $\theta = 90°$ (save for very short wave lengths). Moreover for a given wave length the variation of intensity with the position of the head is more marked for short distances than for long, and for a given distance is more noticeable for short than for long wave lengths. For the possible practical applications of these facts the reader should consult the article of Stewart above referred to.

The foregoing discussion has at least two points of merit. It gives quantitative values for varied wave lengths and distances and thus makes the phenomena concrete and more readily understood and utilizable. In the second place, it suggests the possibility of using a sphere for the location of a receiver in the problem of sound intensity measurement. This would prevent the distortion of the undisturbed intensity distribution by such an irregularly shaped body as an ordinary microphone and substitute a known modification. This has actually been done by Ballantine.[1]

It should, of course, be noted that when sound waves meet a rigid obstacle of dimensions small compared with the wave length they are scattered in all directions. We shall not examine the

[1] S. Ballantine, Phys. Review, 32, 988, 1928.

scattered wave analytically,[1] but it is worth while to summarize briefly the results of calculations on this point. These show that the *amplitude* of the scattered waves at any point distant from the obstacle is directly proportional to the *volume* of the obstacle and inversely proportional to the square of the *wave length*. The intensity of the scattered sound therefore varies inversely as the fourth power of the wave length, analogously to the optical law of the scattering of light by very small particles. Lord Rayleigh has pointed out an acoustical illustration in the so-called "harmonic echoes." If a compound musical note is sounded near some diffracting obstacles like a grove of trees, the intensity of the octave as compared with the fundamental in the scattered sound is found to be many times what it was in the original note. The scattered sound may thus appear to be raised an octave in pitch.

1·12. Introduction to the Fundamental Theory of Acoustic Waves in Fluids.—General Wave Equation.—In this book the exposition of the theory does not proceed in an orderly manner from beginning to end, but rather is presented where actually used. Such a plan is impracticable unless the reader may be assumed to have in his possession the derivation of the most important fundamental equations. This is the purpose of the next five sections of this chapter. The reader need not digest them at once but may acquaint himself with the assumptions and processes involved as he finds references to the fundamental equations derived herein.

Whenever a disturbance is created at any point in a compressible fluid there is a propagation of that disturbance throughout the fluid. We desire to study the characteristics of this propagation. For this purpose the following symbols must be introduced:

x, y, z, coordinates of a particle of the medium,
u, v, w, component velocities of a particle of the medium,
ρ, density,
p, pressure in the medium,
c, velocity of propagation of the disturbance,
s, the condensation, defined by the relation $s = \frac{\rho - \rho_0}{\rho_0}$
$= \frac{\delta\rho}{\rho_0}$, where ρ_0 is the constant mean density at any point,
φ, the velocity potential,
ξ, η, ζ, the component particle displacements along x, y, z axes.

[1] See Lamb, Dynamical Theory of Sound, 1925, p. 244.

Of the above quantities all except c are functions of x, y, z and t. We have now to define the velocity potential, the most important single quantity in the study of the irrotational motion of fluids. This is done in the following equation:[1]

$$u = \frac{\partial \varphi}{\partial x}, \qquad v = \frac{\partial \varphi}{\partial y}, \qquad w = \frac{\partial \varphi}{\partial z}. \tag{1·8}$$

Let us consider a small volume element of the fluid. The difference between the efflux and influx of the medium in this element is equal, by the so-called principle of continuity, to the time rate of growth of mass in the element.

The most simple method of obtaining the mathematical expression of this principle is by the consideration of the elemental parallelepiped of dimensions Δx, Δy, Δz. By considering the influx and efflux through each pair of faces respectively, the difference between the latter and the former for the whole cube is found to be:

$$-\left\{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right\} \Delta x \Delta y \Delta z.$$

Moreover, the rate of growth of mass in the cube is clearly $\partial\rho/\partial t \times \Delta x \Delta y \Delta z$. Equating these two quantities we get the following relation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0. \tag{1·9}$$

If in this equation we make the substitution $\rho = \rho_0(1 + s)$, we have:

$$\rho_0 \frac{\partial s}{\partial t} + \rho_0(1 + s)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \rho_0\left(u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} + w\frac{\partial s}{\partial z}\right) = 0.$$

Now in acoustics the condensation is a very small quantity com-

[1] In the definition of φ there is no general agreement as to sign, some authors, as for example Lamb and Crandall, preferring the negative sign (i.e., $u = -\frac{\partial \varphi}{\partial x}$, etc.), while others, including Rayleigh, use the positive sign as above. The former usage serves merely to emphasize an analogy to the electrostatic and gravitational potentials, which it does not seem necessary to the present authors to press unduly.

Naturally our use of φ implies its existence, which in turn implies that the principal motions of a fluid of concern in acoustics are irrotational. (See Rayleigh, Vol. II, Chap. XI.)

pared with unity. Thus, for example, atmospheric pressure is approximately 10^6 dynes per cm^2, and the variation in pressure from the mean in a sound having the intensity of conversational speech is, roughly, 10^{-1} dyne per cm^2. For an adiabatic change it is well known that $dp/d\rho = \gamma p/\rho$, where for air $\gamma = 1.41$. Hence $s = \delta\rho/\rho_0$ is of the order of 10^{-7}. It can also be shown that u, v, w are usually of the order of 10^{-1} to 10^{-2} cm per sec. Moreover, acoustical wave lengths are so long that u, v, w and s change very little with x, y, z. Hence $\partial u/\partial x$, $\partial s/\partial x$, etc., are very small quantities. We can therefore neglect terms like $s(\partial u/\partial x)$ and $u(\partial s/\partial x)$ in comparison with $\partial u/\partial x$ and to this approximation the continuity equation becomes:

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (1\cdot10)$$

Substituting from (1·8) we have:

$$\frac{\partial s}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0,$$

or

$$\frac{\partial s}{\partial t} + \nabla^2 \varphi = 0 \qquad (1\cdot11)$$

in the more compact notation.

To replace (1·11) by an equation in which φ is the only independent variable we must have recourse to the hydrodynamical equations of motion, which are: [1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

and two similar equations for v and w, wherein the left-hand side represents the acceleration of the medium in the x direction and the right-hand side the force per unit mass, and there are no external forces acting. From what has been said above about the orders of magnitude of these quantities we can neglect $u(\partial u/\partial x)$, etc., compared with $\partial u/\partial t$, etc. Our equations of motion then take the simpler form:

[1] See, for example, Lamb, Dynamical Theory of Sound, 2d Edition, p. 201. We have been discussing continuity. As will be noticed throughout this book, a useful differential equation is obtained by considering not only *continuity* but also *motion*. Consistently, therefore, we will combine these two views of the state of affairs. This is illustrated by our present recourse to the equations of motion.

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

If we substitute into these equations $u = \partial\varphi/\partial x$, etc., multiply the three equations by dx, dy, dz respectively and add, we have

$$\frac{\partial}{\partial t}d\varphi = -\frac{1}{\rho}dp,$$

or integrating:

$$\frac{\partial \varphi}{\partial t} = -\int \frac{dp}{\rho}, \tag{1·12}$$

where the integration is to be thought of as extending from a point in the wave where the velocity potential and excess pressure have values different from zero to a point where they are both zero. Since ρ changes but little it will be approximately correct to remove it from under the integral sign, calling it ρ_0, the mean density. Then $\int dp$ reduces simply to the excess pressure, and if we write it δp to avoid confusion, we have the approximate relation

$$\frac{\partial \varphi}{\partial t} = -\frac{\delta p}{\rho_0}. \tag{1·13}$$

Now there is a relationship between the pressure and density of the fluid. Ignoring for the present its exact nature let us set $\delta p = c^2\delta\rho$. But $\delta\rho = \rho_0 s$ by the definition of condensation, and hence there follows the important relationship

$$\delta p = c^2\rho_0 s. \tag{1·14}$$

Moreover (1·13) now becomes

$$\frac{\partial \varphi}{\partial t} = -c^2 s, \tag{1·15}$$

which is a *general equation* connecting velocity potential and condensation. If we substitute from (1·15) into (1·11) we have finally

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2\nabla^2\varphi, \tag{1·16}$$

which is the familiar equation of wave motion. The general solution of (1·16) shows [1] that the influence of any value of φ is propagated with velocity c, and that therefore the velocity of any type of wave motion may be regarded as c. In what follows it will be shown specifically that this is true for plane waves and spherical waves, but it is to be remembered that c has the general significance stated.

1·13. Plane Waves.—The equations of the preceding section are perfectly general, their validity being limited only by the continuity of the medium and certain approximations requiring waves of small amplitude. Of course, the medium is never, strictly speaking, a continuous one, but a remark concerning the case of a gas will make clearer the justification of the assumption of practical continuity. From the kinetic theory of gases it follows that for a diatomic gas the velocity c of sound is 0.68 of the root mean square molecular velocity, u.[2] For a monatomic gas the factor is 0.74. This is true irrespective of temperature and density. Since c and u are of the same order of magnitude, the condition of continuity is sufficiently well met in gases.[3] In liquids and solids we are dealing with molecules which are closely packed and governed by molecular forces. Here the condition of continuity is closely approximated. If φ is a function of x and t only we have the familiar problem of *plane waves*, which we shall now study in detail. The wave equation (1·16) then reduces to [4]

[1] See Jeans, Electricity and Magnetism, 1925, p. 521.

[2] See Jeans, Dynamical Theory of Gases, 1916, pp. 374–377.

[3] For very high frequencies it is evident that the study of the passage of a sound wave through a gas is going to involve kinetic theory considerations. In particular it will be necessary to take into account the relatively slow rate of exchange of energy between the translational movement of the molecules and their internal degrees of freedom. This has been done by some recent investigators. (K. F. Herzfeld and F. O. Rice, Phys. Rev., 31, 691, 1928; D. G. Bourgin, Phil. Mag., 7, 821, 1929; Phys. Rev., 34, 521, 1929.) It is shown that for very high frequencies the velocity of propagation in gases should increase slightly with the frequency. Earlier experimental work by G. W. Pierce (Am. Acad. Sci., 60, 271, 1929) indicated the existence of such an effect but more recent work by the same investigator shows that in air at least the effect is not measurable. Bourgin states that this conclusion is not in conflict with his kinetic theory for the propagation of sound in mixed gases. Theoretical investigations of this kind may prove of importance in connection with supersonics (see Sec. 10·9).

[4] For time derivatives we shall hereafter use the dot notation, i.e.,

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t}; \qquad \ddot{\varphi} = \frac{\partial^2 \varphi}{\partial t^2}, \text{ etc.}$$

$$\ddot{\varphi} = c^2 \frac{\partial^2 \varphi}{\partial x^2}, \tag{1·17}$$

of which the general solution[1] is

$$\varphi = f(ct - x) + F(ct + x), \tag{1·18}$$

wherein f and F are arbitrary functions. The first term on the right-hand side represents a wave moving in the positive x direction. Thus the value of this term at $x = x_0$ for $t = 0$, is $f(-x_0)$, and at $x = ct_0 + x_0$ at the later time $t = t_0$, will be $f[ct_0 - (ct_0 + x_0)] = f(-x_0)$. That is, the disturbance at x_0 has travelled to $x_0 + ct_0$ in the time t_0. This means progress in the positive x direction with velocity c. Similar reasoning will show that $F(x + ct)$ represents a wave propagated in the negative x direction.

To get equation (1·16) we placed $c^2 = \delta p/\delta\rho = dp/d\rho$. Therefore we see that independently of the functional relationship of p and ρ, $\sqrt{dp/d\rho}$ gives the velocity of a plane wave in a fluid medium. From the fact that $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ are both zero we see also that this wave is a longitudinal wave, i.e., the displacements of the medium are in the direction of propagation.

If we confine our interest to the waves travelling to the right and assume for simplicity that they are simple harmonic waves, we have

$$\varphi = f(ct - x) = A \cos \frac{2\pi}{\lambda}(ct - x),$$

or

$$\varphi = A \cos k(ct - x), \tag{1·19}$$

where $k = 2\pi/\lambda$ and $\lambda =$ wave length or distance between two successive maxima. The frequency ν of the simple harmonic wave is the number of waves that pass any point in one second. It is given by $\nu = c/\lambda$. The quantity $\omega = 2\pi\nu$ is often used in texts on acoustics and electric wave theory, being sometimes called the "speed." (Obviously $k = \omega/c$.) A is the amplitude of φ. The spatial variation of φ is controlled by kx. From the eqs. (1·8) and (1·19) we have

$$u = \frac{\partial \varphi}{\partial x} = kA \sin k(ct - x), \tag{1·20}$$

[1] The solution of eq. (1·17) need not be given here. It can be found in almost any text on differential equations. See, for example, Cohen: Differential Equations, p. 233. Also Rayleigh: Theory of Sound, Vol. I, Art. 144.

and from (1·15)

$$\dot{\varphi} = -c^2 s = -Akc \sin k(ct - x), \tag{1·21}$$

so that

$$s = \frac{k}{c} A \sin k(ct - x). \tag{1·22}$$

Moreover for the excess pressure we have $\delta p = c^2 \rho_0 s$ (eq. (1·14)). Whence

$$\delta p = kc\rho_0 A \sin k(ct - x), \tag{1·23}$$

wherein we shall for future convenience replace δp by p and always consider the latter as the *excess pressure*.

From the above we conclude that u, p and s are in phase with one another; that is, when the particle velocity is positive, the excess pressure and condensation are likewise positive. That this is not true of the particle displacement, ξ, may be seen from the following. We have from (1·20)

$$\dot{\xi} = u = kA \sin k(ct - x), \tag{1·24}$$

whence, since we omit any value of ξ independent of the time,

$$\xi = -\frac{A}{c} \cos k(ct - x) = -\frac{A}{c} \sin k\left(ct - x + \frac{\pi}{2k}\right). \tag{1·25}$$

Equation (1·25) thus shows that the displacement is $\pi/2$ or 90° in phase ahead of the velocity, condensation and pressure. Incidentally, by differentiating ξ with respect to x and comparing with (1·22), we note that

$$s = -\frac{\partial \xi}{\partial x}. \tag{1·26}$$

This important relation will be of use later. Physically it means that if ξ increases in the *positive* x direction we get a *rarefaction* (i.e., s is negative); while decreasing ξ in the positive x direction corresponds to a *condensation*. It should be noted that if ξ is measured always in the direction of propagation, as we shall later find it desirable to do (see Sec. 1·17), then for a wave in the negative x direction eq. (1·26) will become $s = +\partial\xi/\partial x$; i.e., we shall use the *minus* or *plus* sign according as we are referring to propagation in the positive or negative x direction.

Since *any* function of $(ct - x)$, say $f(ct - x)$, obeys the wave equation, i.e.,

$$\ddot{f} = c^2 \frac{\partial^2 f}{\partial x^2},$$

it follows that not only does φ satisfy eq. (1·17) but that we may substitute for φ therein ξ, u, p, s, as we choose (recalling that here p means *excess pressure*). From the fact that $F(x + ct)$ is a solution of the wave equation, it also follows that all the preceding results of this section hold likewise for a wave progressing in the negative x direction, and hence for the sum of any number of waves in the positive and negative directions, assuming that the displacement and particle velocity are measured in the same direction for all. We have already mentioned the change in sign introduced into eq. (1·26) by the measurement of positive ξ in the direction of propagation. This will occur only in equations containing displacement and particle velocity.

We have seen that the general expression for the velocity of a wave in a fluid is $c = \sqrt{dp/d\rho}$. In texts on heat it is shown that if in a gas we assume isothermal change, we have $dp/d\rho = p/\rho$ (i.e., Boyle's law holds), whence

$$c = \sqrt{\frac{p}{\rho}}. \tag{1·27}$$

But if the change is adiabatic (and this is indeed the case for the rapid expansions and contractions accompanying the progress of a sound wave in a gas), then $dp/d\rho = \gamma p/\rho$, where γ = ratio of specific heat at constant pressure to specific heat at constant volume. For this case then,

$$c = \sqrt{\frac{\gamma p}{\rho}}. \tag{1·28}$$

In the case of a liquid we must introduce the elasticity. Thus if we call ΔV the change in volume of a given mass of liquid corresponding to a change in pressure Δp, then the ratio

$$\frac{\Delta p}{-\dfrac{\Delta V}{V}}$$

is found to be practically a constant if Δp is not large. This constant is called the "volume elasticity" or "bulk modulus," and is here denoted by E. From the definition of density we may write $\Delta V/V = -\Delta\rho/\rho$, or in the limit $dV/V = -d\rho/\rho$; whence since $V dp/dV = -E$, we have

$$\frac{dp}{d\rho} = \frac{E}{\rho}, \tag{1·29}$$

from which there results for the velocity of the compressional wave in the liquid

$$c = \sqrt{\frac{E}{\rho}}. \tag{1·30}$$

Although the discussion in this section refers particularly to plane waves, it should be emphasized that eqs. (1·28) and (1·30) are true for any type of compressional wave whatever (i.e., spherical or cylindrical) since their derivations have in no wise limited the meaning of c, which has already been shown to be general.

1·14. Spherical Waves.—Returning to our general equation (1·16),

$$\ddot{\varphi} = c^2\nabla^2\varphi,$$

let us assume spherical symmetry about a point. That is, φ will be assumed to depend only on t and the distance r from this point, which may be conceived as a point source of sound and made for convenience the pole of polar coordinates r, θ and ψ. The disturbance in such a case is propagated as a spherical wave. In polar coordinates it may be shown [1] that $\nabla^2\varphi$ takes the form:

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\psi^2}, \tag{1·31}$$

where $x = r\sin\theta\cos\psi$; $y = r\sin\theta\sin\psi$; $z = r\cos\theta$. In the case of symmetry assumed, the above expression reduces to

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r},$$

or

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi). \tag{1·32}$$

Therefore the general wave equation (1·16) becomes

$$\ddot{(r\varphi)} = c^2\frac{\partial^2(r\varphi)}{\partial r^2}. \tag{1·33}$$

From the solution of eq. (1·17) for the case of plane waves it follows that the solution of (1·33) for the case of spherical waves is

$$r\varphi = f(ct - r) + F(ct + r),$$

or

$$\varphi = \frac{1}{r}[f(ct - r) + F(ct + r)]. \tag{1·34}$$

[1] See, for example, Houstoun, Introduction to Mathematical Physics, p. 21.

In this expression $(1/r)f(ct - r)$ represents a wave expanding or diverging from the point source, while $(1/r)F(ct + r)$ represents a contracting wave converging to the point. The velocity of each of these waves is $c = \sqrt{dp/d\rho}$ as before. We can readily find the expressions for s, ξ and p (the *excess* pressure) for a spherical wave if that for φ is known. It might be emphasized here that the great importance of the velocity potential in acoustics is that all the important quantities may be conveniently obtained as soon as one has the velocity potential. Thus from our previous work ((1·15) etc.) we have established the relations:

$$s = -\frac{1}{c^2}\dot{\varphi},$$

$$\dot{\xi} = u = \frac{\partial \varphi}{\partial r},$$

$$p = c^2\rho_0 s = -\rho_0\dot{\varphi}.$$

To develop useful expressions for these quantities we must assume a particular type of wave. As before we shall postulate a harmonic wave and for convenience shall employ the complex notation.[1] Thus

$$\varphi = \frac{A}{r}e^{ik(ct-r)}, \tag{1·35}$$

where A denotes the constant *amplitude* factor. From this:

$$s = -\frac{1}{c^2}\dot{\varphi} = -\frac{ikA}{cr}e^{ik(ct-r)}. \tag{1·36}$$

Since

$$\dot{\xi} = \frac{\partial \varphi}{\partial r} = -\left(\frac{1}{r} + ik\right)\frac{A}{r}e^{ik(ct-r)}, \tag{1·37}$$

it follows that

$$\xi = -\left(\frac{1}{r} + ik\right)\frac{A}{irkc}e^{ik(ct-r)}, \tag{1·38}$$

of which the real part is

$$\xi = -\left[\cos k(ct - r) + \frac{1}{kr}\sin k(ct - r)\right]\frac{A}{rc}$$

$$= -\frac{A}{rc\cos\theta}\cos\left[k(ct - r) - \theta\right], \tag{1·39}$$

wherein

$$\tan\theta = \frac{1}{kr} = \frac{\lambda}{2\pi r}. \tag{1·40}$$

[1] See, for example, Lamb: Dynamical Theory of Sound, 2d ed., p. 53.

Since $p = c^2\rho_0 s$, we have

$$p = -\frac{ikcA\rho_0}{r} e^{ik(ct-r)}, \tag{1·41}$$

of which the real part is

$$p = \rho_0 \frac{kcA}{r} \sin k(ct - r). \tag{1·42}$$

From eqs. (1·39) and (1·42) we see that p and ξ are no longer different in phase by 90° as in the case of plane waves. The additional phase difference θ increases with the wave length for constant r and decreases with r for constant λ. However, p and s are in phase as before.

These last three sections have presented the derivation of the general equations which will be used constantly in the text that follows. Those most commonly met are (1·13), (1·14), (1·15), (1·16) and (1·26). In the cases of plane and spherical waves, the equations give the relations, both in magnitude and phase, of the various physical quantities such as pressure and particle velocity. These will be repeatedly used in the chapters that follow. Cylindrical waves are omitted from consideration because in practice they can usually be treated as plane and also because the theory would be burdened by non-useful complications if they were inserted.

1·15. Energy Content of an Acoustic Wave.—Intensity.—If we denote the excess pressure by p, from Section 1·13 we have the relation $p = \rho_0 c^2 s$, where ρ_0 is the mean equilibrium density, c the velocity of sound and s the condensation. It is now desired to find the kinetic and potential energies associated with the wave. The former, by definition, is given by

$$K.E. = \tfrac{1}{2}\rho_0 \int\int\int \dot{\xi}^2 dV, \tag{1·43}$$

where the integration is taken over the whole volume of the fluid being considered. To get the potential energy per unit volume consider a small element of volume ΔV. Under excess pressure p, this contracts to $\Delta V(1 - ds)$ for a change in the condensation ds, if terms of the order of $\Delta V \cdot ds^2$ and $\Delta V \cdot sds$ and higher are neglected. An amount of work is then done equivalent to $p \cdot \Delta V \cdot ds$. Consequently the work done on the element while the condensation

varies from o to s is given by:

$$\int_0^s p\Delta V ds = \Delta V \int_0^s \rho_0 c^2 s ds = \tfrac{1}{2}\Delta V \rho_0 c^2 s^2,$$

which reduces to $\frac{1}{2}\rho_0 c^2 s^2$ per *unit* volume. Hence the total potential energy is

$$P.E. = \tfrac{1}{2}\rho_0 c^2 \int\int\int s^2 dV. \qquad (1\cdot44)$$

Now in the case of a plane progressive wave where we have, for example, $\xi = f(ct - x)$, it follows that $\dot{\xi} = cf'(ct - x)$ while $\partial\xi/\partial x = -f'(ct - x)$, where the primes indicate differentiation with respect to the whole argument of the function. But from Section 1·13 we also recall (eq. (1·26)) $\partial\xi/\partial x = -s$. It therefore follows that for a plane wave

$$\dot{\xi} = u = cs. \qquad (1\cdot45)$$

Comparison of (1·43) and (1·44) in this case shows that for plane waves the energy content is half potential and half kinetic. We must take care to note that this is not a general condition. For example, it is not true of a diverging spherical wave, except at great distances from the source.

The *intensity* of a sound wave (in the case of plane and spherical waves, which are the ones we shall deal with exclusively) is defined as the average rate of flow of energy per unit area normal to the direction of propagation. It is thus the power transmission per unit area, and for a plane wave is obviously the same as the average energy content per unit volume (i.e., energy density) multiplied by the velocity of sound. For a plane harmonic wave where $\xi = A\cos k(ct - x)$, we have $\dot{\xi}^2 = A^2k^2c^2\sin^2 k(ct - x)$, whence from the previous paragraph the average energy density is

$$\overline{W} = \frac{\rho_0 \int_x^{x+n\lambda} A^2k^2c^2 \sin^2 k(ct - x)dx}{n\lambda}, \qquad (1\cdot46)$$

where λ is the wave length and n any integer. Carrying through the integration there results

$$\overline{W} = \tfrac{1}{2}A^2k^2c^2\rho_0, \qquad (1\cdot47)$$

wherein we note that the result is, as it should be, independent of t. Now we have from the above for the maximum particle

velocity $\dot{\xi}_{max} = Akc$, whence there follows from (1·45) $Ak = s_{max}$. Therefore we get an alternative form for the average energy density in terms of the maximum condensation, viz.

$$\overline{W} = \tfrac{1}{2}\rho_0 c^2 s^2_{max}. \tag{1·48}$$

A still more common form for $\overline{W}$ is in terms of the maximum excess pressure, viz., $p_{max} = \rho_0 c^2 s_{max}$. Thus, substituting, there results

$$\overline{W} = \tfrac{1}{2}\frac{p^2_{max}}{\rho_0 c^2}, \tag{1·49}$$

whence the intensity becomes

$$I = \tfrac{1}{2}\frac{p^2_{max}}{\rho_0 c}. \tag{1·50}$$

Now it can be shown (using eq. (1·23) in Sec. 1·13) that (1·50) is also the time average value of the expression $p^2/\rho_0 c$. If a procedure somewhat similar to the above is applied to spherical waves, the same expression is found for the intensity.[1] It is thus generally customary to think of intensity as always proportional to the mean square of the excess pressure, and as such the term is a very valuable one with reference to acoustic instruments, most of which are operated by pressure (e.g., diaphragms and the like). It is important to note that when the intensity is expressed in the above form the frequency is not involved.

Recently there has come into extensive use a logarithmic unit for denoting *difference* in levels of sound intensity.[2] Thus if I_1 and I_0 are two different intensities being compared, we may say that the two differ by α "bels" where

$$\alpha = \log_{10} I_1/I_0.$$

A unit of more convenient size in telephone engineering is the "decibel" (db) which is one-tenth of a bel. We shall have occasion to refer to this unit in connection with the acoustics of audition (Sec. 9·7).

1·16. Variation of Energy Content with Frequency and Amplitude.—In the case of plane waves it was shown in Section 1·13 that for a displacement given by $\xi = -A/c \cdot \cos k(ct - x)$ the corresponding excess pressure is $p = k\rho_0 cA \sin k(ct - x)$. Since k

[1] See problem 18 of this chapter and Sec. 3·2 of Chap. III.

[2] See, for example, Fletcher, Speech and Hearing, 1928, p. 69.

$= 2\pi/\lambda$, it follows that $p = 2\pi\rho_0 cA/\lambda \cdot \sin k(ct - x)$. Hence the excess pressure has a maximum value which varies directly as the displacement amplitude and inversely as the wave length or directly as the frequency of the wave. It therefore follows from the preceding section that the potential energy of a plane wave is directly proportional to the square of the product of the displacement amplitude and frequency. The intensity of the resultant of two plane waves of the same frequency is proportional to the square of the product of the resultant displacement amplitude and the frequency.

1·17. Phase of a Wave.—Influence of Reflection on Phase.— It will be found convenient to make some convention with regard to the relation between the direction of positive displacement in a wave and the direction of propagation. In this text it will be assumed that the particle displacement is *positive* if in the direction of propagation and *negative* if in the *reverse* direction. We have now to consider the important matter of the *phase* of a wave. For this purpose let us consider a plane harmonic wave with displacement written in complex form, viz.

$$\xi = Ae^{i(\omega t - kx)} = A[\cos(\omega t - kx) + i \sin(\omega t - kx)],$$

where $k = \omega/c$, as usual. The student should at this point familiarize himself with the complex notation (see note on Sec. 1·14), since it is of great value in the more complicated developments of the subject. When written in the above form, the quantity $\omega t - kx$ is called the "phase" of the displacement, provided the amplitude A is real. If the amplitude is complex it can always be written in the form $A = A_0 e^{i\epsilon}$, where A_0 is real and the displacement takes the form

$$\xi = A_0 e^{i(\omega t - kx + \epsilon)},$$

and the phase is now $\omega t - kx + \epsilon$. The quantity ϵ is sometimes called the *initial phase*. If we have two plane harmonic waves of the same frequency progressing in the same direction with the same velocity, they will clearly be of the *same phase* if

$$\omega t - kx + \epsilon_1 = \omega t - kx + \epsilon_2,$$

or $\epsilon_1 = \epsilon_2$. They will then have at any point maximum displacements at the same time. With regard to waves traveling in opposite directions the situation is more complicated. Nevertheless we

can say that at any *given* point two such waves are in the same phase if they attain their maximum displacements there at the same time, these displacements being measured, of course, in opposite directions, in accordance with our initial convention.

Let us then consider the effect on the phase of the reflection of sound at the boundary between two fluid media. We shall here assume normal incidence, for simplicity. The incident wave (from left to right) may have its displacement denoted by

$$\xi_i = \xi_0 e^{i(\omega t - kx)}. \tag{1·51}$$

The reflected wave displacement after the establishment of a "steady state" is

$$\xi_r = \xi_1 e^{i(\omega t + kx + \epsilon)}, \tag{1·52}$$

where the amplitudes ξ_0 and ξ_1 are real and positive. Now suppose that at the boundary the resultant displacement is always *zero*, corresponding to the transition *to* an *infinitely* dense medium which acts like a rigid wall. Then we have, recalling that ξ_r is measured to the left,

$$\xi_i - \xi_r = 0, \tag{1·53}$$

which leads to

$$\xi_0 = \xi_1 e^{i\epsilon}, \tag{1·54}$$

if the boundary is taken at $x = 0$. Separating the real and imaginary parts, the above equation yields

$$\xi_1 \sin \epsilon = 0, \tag{1·55a}$$

$$\xi_0 = \xi_1 \cos \epsilon. \tag{1·55b}$$

From (1·55*a*) we infer that $\epsilon = n\pi$ where n is any integer; effectively we are here concerned only with $n = 0$ or 1. For $\epsilon = 0$ we have, from (1·55*b*),

$$\xi_0 = \xi_1, \tag{1·56a}$$

while for $\epsilon = \pi$ we should have

$$\xi_0 = -\xi_1. \tag{1·56b}$$

But this is inconsistent with eq. (1·53) and the fact that ξ_0 and ξ_1 are real and positive. Hence eq. (1·56*a*) alone applies here, producing zero displacement at $x = 0$ at all times. The reflection occurs, then, with $\epsilon = 0$ or *without* change in phase.

Next consider the situation in which sound passes from a dense medium into an *infinitely* rare medium. The second medium will

not support pressure changes. Hence at the boundary the resultant excess pressure or condensation will be always zero. Now $s_i = -\partial\xi_i/\partial x$ and $s_r = +\partial\xi_r/\partial x$ [1] and hence at $x = 0$

$$s = -\frac{\partial}{\partial x}(\xi_i - \xi_r) = 0. \tag{1·57}$$

Proceeding as above this yields

$$\xi_0 + \xi_1 e^{i\epsilon} = 0, \tag{1·58}$$

leading again to $\epsilon = 0$ or π. But here $\epsilon = 0$ gives $\xi_0 = -\xi_1$, while $\epsilon = \pi$ gives $\xi_0 = \xi_1$. Hence we now say that the reflection takes place with a phase change of π.

A somewhat simpler view of the preceding may be obtained if we use complex amplitudes [2] and omit the introduction of the phase into the exponent. Thus let the incident and reflected displacements now be denoted by

$$\xi_i = (\xi_0' + i\xi_0'')e^{i(\omega t - kx)}, \tag{1·59}$$

$$\xi_r = (\xi_1' + i\xi_1'')e^{i(\omega t + kx)}, \tag{1·60}$$

where

$$\xi_0' + i\xi_0'' = \xi_0 \qquad \text{and} \qquad \xi_1' + i\xi_1'' = \xi_1.$$

Then if the displacement at $x = 0$ is always zero we have

$$(\xi_0' + i\xi_0'') = \xi_1' + i\xi_1'', \tag{1·61}$$

whence $\xi_0' = \xi_1'$, $\xi_0'' = \xi_1''$ immediately, indicating that the reflection takes place without change of phase. We can also look at it in this wise. The ratio of two complex numbers such as ξ_1 and ξ_0 is a complex number which can always be written in the form

$$\frac{\xi_1}{\xi_0} = A_0 e^{i\epsilon}, \tag{1·62}$$

where A_0 is real and positive and equal to $|\xi_1|/|\xi_0|$ and ϵ denotes the difference in phase of ξ_1 and ξ_0. Now if $\xi_1 = \xi_0$ we have $A_0 e^{i\epsilon} = 1$ or $A_0 \cos\epsilon = 1$ and $A_0 \sin\epsilon = 0$. Then $A_0 = 1$ and $\epsilon = 0$ corresponding to the case just discussed, i.e., reflection without

[1] See Section 1·13, eq. (1·26), and accompanying discussion.

[2] We shall find the use of complex amplitudes very convenient in further work on transmission problems (see Chap. III).

change in phase. In the second case in which the excess pressure at the boundary ($x = 0$) is zero, we have $\xi_1 = -\xi_0$, whence $A_0 e^{i\epsilon} = -1$ leading to $A_0 = 1$ and $\epsilon = \pi$, corresponding to reflection with phase change of π.

The treatment of this section is idealized in the assumption of infinitely dense and infinitely rare media. We now find it necessary to treat the general case of normal reflection at the boundary of *any* two fluid media.

1·18. Normal Reflection at Boundaries in Gaseous Media.— The boundary conditions that must be satisfied here are that the particle velocities normal to the interface and the excess pressures on the two sides are equal in each case. Following the notation of Section 1·17, let the displacements in the incident, reflected and transmitted waves after the establishment of a steady state be, respectively,

$$\xi_i = \xi_0 e^{i(\omega t - k_1 x)}$$
$$\xi_r = \xi_1 e^{i(\omega t + k_1 x)}$$
$$\xi_t = \xi_2 e^{i(\omega t - k_2 x)},$$

where $k_1 = \omega/c_1$, $k_2 = \omega/c_2$, while c_1 is the velocity of sound in the first medium, c_2 is the velocity of sound in the second medium, and ξ_0, ξ_1 and ξ_2 are complex quantities whose relations of magnitude and phase we desire to investigate. Let the equilibrium densities of the two media be ρ_{01} and ρ_{02} respectively. If the boundary is taken at $x = 0$, the conditions become

$$\dot{\xi}_i - \dot{\xi}_r = \dot{\xi}_t, \text{ at } x = 0 \qquad (1\cdot63a)$$

$$\rho_{01} c_1^2 (s_i + s_r) = \rho_{02} c_2^2 s_t, \text{ at } x = 0, \qquad (1\cdot63b)$$

where s_i, s_r and s_t are respectively the incident, reflected and transmitted condensations, and we recall that ξ_r is measured in the negative x direction, i.e., the direction in which the reflected wave moves. Performing the indicated differentiations and substitutions (recalling that $s_i = -\partial\xi_i/\partial x$, while $s_r = +\partial\xi_r/\partial x$), the equations (1·63*a*) and (1·63*b*) finally reduce to

$$\xi_0 - \xi_1 = \xi_2 \qquad (1\cdot63c)$$

$$\rho_{01} c_1 (\xi_0 + \xi_1) = \rho_{02} \xi_2. \qquad (1\cdot63d)$$

On eliminating ξ_2 between these two there results for the ratio of

the reflected and incident complex displacement amplitudes at the boundary

$$\frac{\xi_1}{\xi_0} = \frac{\rho_{02}c_2 - \rho_{01}c_1}{\rho_{02}c_2 + \rho_{01}c_1}. \tag{1·64}$$

It is worthy of note that (1·64) is a general equation holding for fluids in general. It may be specialized to apply to gases only by the substitution $c_1 = \sqrt{\gamma_1 p/\rho_{01}}$, etc. Then (1·64) becomes

$$\frac{\xi_1}{\xi_0} = \frac{\sqrt{\gamma_2\rho_{02}} - \sqrt{\gamma_1\rho_{01}}}{\sqrt{\gamma_2\rho_{02}} + \sqrt{\gamma_1\rho_{01}}} = \frac{\gamma_2 c_1 - \gamma_1 c_2}{\gamma_2 c_1 + \gamma_1 c_2}. \tag{1·65}$$

If, as is often the case, $\gamma_1 = \gamma_2$ approximately, (1·65) reduces to the simpler and more common form

$$\frac{\xi_1}{\xi_0} = \frac{\sqrt{\rho_{02}} - \sqrt{\rho_{01}}}{\sqrt{\rho_{02}} + \sqrt{\rho_{01}}} = \frac{c_1 - c_2}{c_1 + c_2}. \tag{1·66}$$

The equation (1·64) yields the phase relationship as well as the amplitude ratio. For setting

$$\frac{\xi_1}{\xi_0} = A_0 e^{i\epsilon},$$

as in the preceding section, so that ϵ is the phase difference between ξ_1 and ξ_0, we see that

$$A_0 e^{i\epsilon} = \frac{\rho_{02}c_2 - \rho_{01}c_1}{\rho_{02}c_2 + \rho_{01}c_1},$$

whence

$$A_0 \cos\epsilon = \frac{\rho_{02}c_2 - \rho_{01}c_1}{\rho_{02}c_2 + \rho_{01}c_1}$$

and $A_0 \sin\epsilon = 0$, so that $\epsilon = 0$ if $\rho_{02}c_2 > \rho_{01}c_1$, while $\epsilon = \pi$ if $\rho_{02}c_2 < \rho_{01}c_1$. In each case

$$A_0 = |\xi_1|/|\xi_0| = \frac{\rho_{02}c_2 - \rho_{01}c_1}{\rho_{02}c_2 + \rho_{01}c_1}.$$

This analysis makes clear the interesting fact that the change of phase on normal reflection of a compressional wave in either a liquid or a gas is always either zero or π. It will be shown later (Sec. 4·5) that this statement is also true for the normal reflection of a compressional wave at the interface of a fluid and a solid and of two solids.

Let us consider two illustrations. Saturated air at 20° C is lighter than dry air at this temperature by about one part in 120. Consequently the reflected amplitude is only about 1/480 part of the incident. Since, as has been shown in Section 1·15, the intensity is proportional to the square of the amplitude, it follows that when sound is incident from dry air on saturated air or the reverse, the reflected intensity is only 1/230,000 part of the incident intensity. To take another example, for dry air at 100° C, $\gamma = 1.41$ and $c = 386$ meters/sec; while for steam at 100° C the same quantities are 1.33 and 405 meters/sec respectively; consequently the amplitude ratio in this case is (using eq. (1·65)) .054 and the intensity ratio .0029.

As might also be expected, computation shows that if the difference between the two media is one of temperature merely, the reflection is very slight.

1·19. Influence of Surroundings on the Energy Emitted by a Vibrator.—A point sinusoidal source of sound is said to be a "constant" source if, the frequency remaining constant, it causes the injection and removal of the same amount of the surrounding medium. A telephone diaphragm when continuously excited may be looked upon as an example of an approximately constant source. Now suppose such a source to be placed at O, very near and to the right of an infinite wall (see again Fig. 1·1). In the hemisphere to the right of the source there will be not only the direct sound from O but also the sound reflected from the wall, equivalent, as we have seen in Section 1·2, to the sound from the image O'. From the discussion on the effect of reflection on phase we further note that both O and O' are in this case sources in the same phase, and since they are separated by a distance small in comparison with the wave length, in the hemisphere to the right of O the displacement produced will be approximately twice that found in the case of O alone, without the wall. In other words, since the intensity or energy flow per unit area varies as the square of the displacement amplitude, there are now the equivalent of two equal sources radiating, in the hemisphere to the right of O, four times the energy of one source alone in the same hemisphere. Moreover the one source alone radiates in two hemispheres, while the two sources, resulting from the action of the wall on the single source, radiate in but one hemisphere. Therefore the total energy radiated in the latter case is twice that in the former.

Precisely the opposite effect is produced when the source is in a dense medium and the wall is replaced by the boundary between this and a much rarer medium. The reason is that the source and image are now opposite in phase and if they are very near together very much less sound energy will be radiated into the hemisphere to the right of O than if there were no boundary. In both cases the effect obviously depends on the distance of the source from the boundary, being less the greater this distance. As will be emphasized in Chapter X, the facts brought out in this paragraph are of great importance in submarine signalling.

This section will serve to emphasize the very important fact that the environment of a source of sound is often very influential in modifying the "load" or energy output of the source.

1·20. Sound Waves of Finite Amplitude.—In Section 1·12 we have derived the wave equation for sound waves. Certain approximations were adopted which made the equation relatively simple and ultimately led to the conclusion that the velocity of a sound wave is constant and equal to $\sqrt{dp/d\rho}$. That this is not true for sound waves of large amplitude is therefore not surprising. Velocities far in excess of this, the "normal" value, have been obtained in explosive waves. That the velocity should vary with the amount of displacement or particle velocity of the medium is to be anticipated. For, consider the case of a wave so long that the variations in velocity and density along it are not noticeable for a considerable distance. At a given point the particle velocity of the medium is u and the neighboring particles have sensibly the same velocity. Consider a small secondary wave to be superimposed. Let its propagation velocity be c in a medium at rest. This velocity at the point of the medium having the velocity u will therefore be $c + u$. Moreover this velocity will depend upon the position along this primary wave that we are considering. If this be true, the velocity of our secondary wave is variable, depending upon the velocity or density of the medium. The force of these remarks would apply equally well to the original wave itself. One might therefore expect that, with waves involving large medium velocities or densities, the velocity of propagation would not be the same for all parts of the wave. This leads to an intricate situation into which we do not wish to enter. For the sake of brevity, we will merely state that abnormal velocities of sound waves have been repeatedly measured. Careful studies of the physical conditions

in such waves have not been made. The nose wave of a fast moving bullet must travel faster than the normal velocity of sound. Lord Rayleigh discussed this in Proc. Roy. Soc. A, 84, p. 247 (1910), offering an explanation as to how it could retain its form. For a discussion of this and other aspects of sound waves of finite amplitude, reference should be made to the original paper. In part, however, he assumes that the forepart of the wave in front of the projectile maintains its permanent regime ("shape") under the influence of viscosity and heat conduction.

Questions and Problems

1. Given a *line* source of sound in the vicinity of an infinite wall, discuss the influence of the wall on the source on the basis of the treatment in Section 1·2. Is there anything peculiar about this case when the length of the line is *not* small compared with the wave length? Explain.

2. Consider a source placed between two parallel walls. Locate the images. Judging from your experience, would an observer be able to recognize these images as distinct?

3. What would be the value of a horizontal (or almost horizontal) large reflector over an open air speaker's stand? At about what elevation above the speaker should this reflector be placed and why?

4. What difficulties will be encountered in the production of sound shadows in water as compared with the problem of "screening" in air?

5. Using the construction indicated in Fig. 1·4, calculate the diameter of the circular disc that will produce the first maximum effect at a distance of one meter from C on the axis, if the frequency is 500 cycles/sec. Find also the diameters for the first minimum and second maximum respectively.

6. Referring again to Fig. 1·4, suppose that a constant source of sound is located at O. Calculate the diameter of the circular screen CD which will give the first maximum intensity at C, if CO = one meter, and the frequency is 500 cycles/sec.

7. In Section 1·4 the pinnae were mentioned as an illustration of the effect of selective reflection. Is this the only factor which enters? Discuss this in connection with the watch experiment alluded to in the same section.

8. Give a physical reason for the necessity of reflection at a change in area of a conduit.

9. Deduce the equations of motion of a perfect fluid used in Section 1·12. That is, show that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ etc.}$$

10. Show by direct substitution that $\varphi = f(ct - x) + F(ct + x)$ is a solution of the wave equation. In particular, prove that $F(ct + x)$ represents a wave progressing in the negative x direction.

11. Verify eq. (1·50) by proving directly that $\overline{p^2} = \frac{1}{2}p^2_{\max}$, where the bar indicates the time average.

12. Prove that when a plane wave of sound is reflected normally in passing from a rare to a dense medium the *condensation* and *excess pressure* are reflected *without* change in phase. Prove that when the passage is from dense to rare the condensation and excess pressure are reflected *with* change in phase of π.

13. Calculate the percentage of sound energy reflected normally in going from air to water.

14. If the observer were moving with the medium (either of two) and the wave is incident at the boundary, what could be said about the perpendicularity of the wave direction and wave front?

15. Stipulate the conditions and sketch the path of sound in a case where it is not reversible, as stated in Section 1·7.

16. Give the details of the solution of eq. (1·17).

17. The solution of eq. (1·17) may be written in any one of three ways:

$$\varphi = f(ct - x) + F(ct + x),$$
$$\varphi = (A' \cos kx + B' \sin kx)e^{i\omega t},$$
$$\varphi = (Ae^{ikx} + Be^{-ikx})e^{i\omega t}.$$

Show that the last two are special cases of the first for simple harmonic waves of one frequency. Discuss the evidence in them for two waves and the differences in significance of A and A'.

18. Starting with the expressions for the kinetic and potential energies as given in eqs. (1·43) and (1·44), apply the fundamental equations of this chapter to show that the time rate of change of the total acoustic energy in a given volume may be expressed in

the form

$$\rho_0 \int\int \dot{\varphi} \frac{\partial \varphi}{\partial n} dS,$$

where $\partial\varphi/\partial n$ is the gradient of φ normal to the surface S bounding the volume, and the integration is to be taken over this surface. Apply this result to plane and spherical waves to deduce the equivalent of eq. (1·50) and eq. (3·24) of Chapter III.

CHAPTER II

Combination of Acoustic Elements

2·1. Acoustic Elements.—The reader is presumably familiar with the use of the term "element" to denote the characteristic components of an electric circuit, i.e., the resistance, inductance and capacitance. Mathematically these elements are merely coefficients in the differential equation connecting electromotive force with current at any instant, but physically we may look upon each of them as defining a distinct and particular activity or behavior of the circuit or part thereof. Thus in a pure resistance electrical energy is dissipated in the form of heat; in a pure inductance the energy is stored in the medium, etc. This treatment of a physical problem in terms of its "elements" is often of great value and will be found in every dynamical theory of physical phenomena. For example, in the next section we shall see that the fundamental elements of a vibrating mechanical system are its inertia, stiffness and mechanical resistance. We should expect that the study of acoustics will be facilitated by the introduction of acoustic elements characterizing the action of each component of an acoustic system. Thus we shall see the acoustic analogues of inertia, stiffness and resistance in the characteristic manner in which the acoustic medium behaves for different sources of sound and differing ways of confining the medium. Indeed every acoustical instrument or system may be looked upon as a combination of acoustic elements. It is the purpose of the present chapter to give an introduction to this important point of view in acoustics.

2·2. Mechanical Oscillations and Resonance.—The vibrations of a thin metal diaphragm will first be considered. Let the effective mass [1] of the diaphragm be m and its area S and let it be acted on

[1] It should be noted that the effective mass of the diaphragm is equal to its actual mass only if all parts of the diaphragm are equally free to vibrate. This is, in effect, the ideal case discussed in this section, which is thus immediately applicable to the oscillations of any mechanical system of one degree of freedom characterized by the appropriate m, f and K_1. The more detailed discussion of the vibrations of a circular diaphragm which is clamped on the periphery is given in Section 8·8. It will there be found that it is possible to replace the vibrations of the diaphragm by those of an equivalent piston with mass and stiffness definitely related to the actual mass and size of the diaphragm.

by an excess pressure p. If we denote the displacement by ξ, and suppose that the elastic restoring force is proportional to ξ, and that there is further a dissipative force proportional to the displacement velocity, the equation of motion becomes

$$m\ddot{\xi} + K_1\dot{\xi} + f\xi = Sp, \tag{2·1}$$

wherein K_1 is the damping (or dissipation) coefficient and f is the stiffness coefficient. If p and ξ are assumed to be simple harmonic in time with frequency $\nu = \omega/2\pi$, then the steady state solution for the resulting oscillation is found by substituting $\xi = \xi_1 e^{i\omega t}$ and $p = p_1 e^{i\omega t}$ into (2·1). Here ξ_1 and p_1 are, in general, complex and are not functions of t. The result is

$$\xi = \frac{Sp}{iK_1\omega + (f - m\omega^2)}, \tag{2·2}$$

or for the more important quantity, the displacement velocity

$$\dot{\xi} = \frac{Sp}{K_1 + i(m\omega - f/\omega)}. \tag{2·3}$$

Rationalizing the denominator we have

$$\dot{\xi} = \frac{Spe^{-i\alpha}}{[K_1^2 + (m\omega - f/\omega)^2]^{1/2}}, \tag{2·4}$$

where $\tan\alpha = (m\omega^2 - f)/K_1\omega$ and α is the *phase angle*, indicating how the velocity lags behind the force. If now we substitute in (2·4) $p = p_1 e^{i\omega t}$ and retain only the real part, we get the equivalent trigonometric equation:

$$\dot{\xi}_{\text{real}} = \frac{Sp_1 \cos(\omega t - \alpha)}{[K_1^2 + (m\omega - f/\omega)^2]^{1/2}}. \tag{2·5}$$

The ratio of the force and the velocity both considered as complex quantities will now be defined as the *mechanical impedance* and denoted by Z. That is, from (2·3),

$$Z = K_1 + i(m\omega - f/\omega). \tag{2·6}$$

Separating this into its real and imaginary parts, we have

$$Z = Z_1 + iZ_2, \tag{2·7}$$

where $Z_1 = K_1$ and is called the *mechanical resistance*, and Z_2

$= m\omega - f/\omega$, the *mechanical reactance*. At the present time no suitable single name has been adopted for either of these quantities.

From eq. (2·5) it is clear that the maximum value of $\dot{\xi}$ occurs for a frequency given by

$$\omega = \sqrt{\frac{f}{m}}, \tag{2·8}$$

and *in this case* the mass is analogous to the self-inductance of an oscillating circuit and the stiffness coefficient is analogous to the *reciprocal* of the capacitance. It is evident that the maximum value of $\dot{\xi}^2$ and hence of the kinetic energy of vibration also occurs at the above so-called *resonance* frequency. This is not true, however, for the displacement. Consulting (2·2) it is seen that the displacement ξ has its maximum for the value of ω which makes

$$K_1^2\omega^2 + (m\omega^2 - f)^2$$

a minimum. This value is

$$\omega = \sqrt{\frac{f}{m} - \frac{K_1^2}{2m^2}}. \tag{2·9}$$

If the effective mass is great and the dissipation or mechanical resistance relatively small, the frequency defined in (2·9) closely approximates the resonance frequency (2·8). The latter may also be called the free oscillation frequency of the system, for it represents the frequency with which the diaphragm would vibrate if disturbed from its equilibrium position and allowed to oscillate freely (assuming no dissipation). It can be shown that if dissipation be taken into account the free oscillation frequency[1] is $1/2\pi \cdot \sqrt{f/m - K_1^2/4m^2}$.

The phase α presents considerable interest. From the relation $\tan\alpha = (m\omega^2 - f)/\omega K_1$, it is seen that if $m\omega^2 - f > 0$, $\pi/2 > \alpha > 0$; while if $m\omega^2 - f < 0$, $\pi > \alpha > \pi/2$. When $\omega = \sqrt{f/m}$, $\alpha = 0$ and the velocity is in phase with the applied force. On the other hand, when $K_1 \doteq 0$, i.e., vanishingly small resistance, $\alpha \doteq \pi/2$. In the latter case the displacement will be found to be either exactly in phase or out by a whole half-period.

The rate of energy dissipation is $K_1\dot{\xi}^2$, whence the application of (2·5), together with the expression $\tan\alpha = (m\omega^2 - f)/K_1\omega$, yields

[1] See Problem 2 at the end of this chapter.

for the time average of this quantity:

$$\overline{\dot{W}} = \overline{\text{energy dissipation rate}} = \frac{S^2 p_1^2 \cos^2 \alpha}{2K_1}$$
$$= \tfrac{1}{2} K_1 \dot{\xi}^2_{\max}. \qquad (2\cdot10)$$

It is of interest to notice that when the velocity is in phase with the force (i.e., $\alpha = 0$), there results maximum energy dissipation. The reader will recall that in pushing a swing to get the greatest effect the pushes should be timed so that the force is a maximum when the *velocity* is a *maximum* and the displacement a *minimum*, i.e., as close to zero as possible. Naturally, the average rate of transfer of energy to the vibrating system by the external force is always equal to the average rate of dissipation.

The important influence that the dissipation or damping factor exercises on the resonance may be seen as follows. Denoting the resonance value of ω by ω_0, we have, from (2·10),

$$\overline{\dot{W}} = \frac{S^2 p_1^2 K_1^2 \omega^2}{2K_1[K_1^2\omega^2 + m^2(\omega^2 - \omega_0^2)^2]},$$

which reduces to

$$\overline{\dot{W}} = \frac{S^2 p_1^2 K_1}{2[K_1^2 + m^2\omega_0^2 x^2]}, \qquad (2\cdot11)$$

where $x = \omega/\omega_0 - \omega_0/\omega$. Then the maximum value of $\overline{\dot{W}}$ occurs for $x = 0$, i.e., the resonance case.

Now we have

$$\frac{d\overline{\dot{W}}}{dx} = -\frac{S^2 p_1^2 K_1 m^2 \omega_0^2 x}{(K_1^2 + m^2\omega_0^2 x^2)^2}. \qquad (2\cdot12)$$

For $x > 0$ (i.e., $\omega > \omega_0$), the derivative is negative, whence as x increases the value of $\overline{\dot{W}}$ falls off at a rate which is *greater* the *smaller* the value of K_1 (other things being equal) and vice versa. But for the resonance case itself ($x = 0$) $\overline{\dot{W}}$ varies inversely as K_1. We may summarize the results (which are presented graphically in the following qualitative diagram, Fig. 2·1) thus: for small dissipation or damping the resonance is sharp and the peak is high; for large damping the resonance is diffuse and the resonance $\overline{\dot{W}}$ is lower. The effect is well known in both acoustic vibrations and electric oscillations.

In most texts on acoustics it is customary at this point to investigate in detail the mechanical vibrations of strings and bars.

And indeed the vibrating string itself furnishes a satisfactory introduction to the theory of vibrating systems. It will suffice here to refer to the many excellent treatments of this topic in the standard texts.[1] The attention of the reader is directed, however, to the problems at the end of Chapter II.

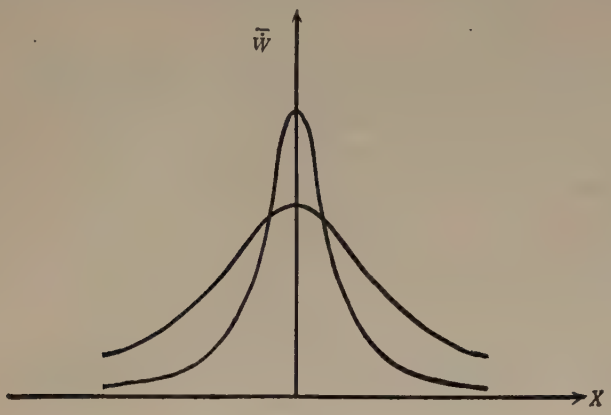

FIG. 2·1.

2·3. Helmholtz Resonator and Acoustic Impedance.—The Helmholtz resonator is an enclosure communicating with the external medium through an opening of small area (see Fig. 2·2). The opening may be flat, as in the figure, or it may be in the form of a neck. In either case it is a simple matter to separate the resonance elements. Inside the resonator there is a volume of gas of magnitude V which is alternately compressed and expanded by the movement of the gas in the opening. It thus provides, so to speak, the *stiffness* element of the system. The gas in the opening moves as a whole and provides the mass or *inertia* element. At the opening, moreover, there is a radiation of sound into the surrounding medium leading to the dissipation of acoustic energy and providing the *dissipation*

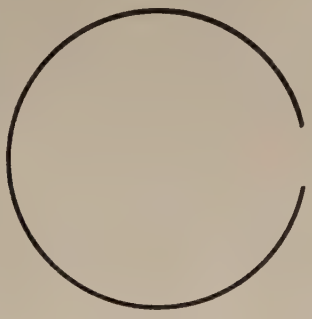

FIG. 2·2.

[1] Rayleigh, Theory of Sound, Vol. I, Chap. VI; Lamb, Dynamical Theory of Sound, 1925, p. 59, 108; Crandall, Vibrating Systems and Sound, 1926, p. 64, 77.

element. To write down the equation of motion of the gas in the resonator we must estimate the magnitudes of the above elements.

If the opening has a neck of length l small in comparison with the wave length, with a cross sectional area S, the mass of gas in the opening is $\rho_0 Sl$. It is customary and convenient to write this in another form by introducing the quantity

$$c_0 = S/l, \tag{2·13}$$

which is called the acoustic *conductivity* of the opening. A more thorough discussion of the conductivity will be given in Section 2·4. For the present we merely substitute it into the mass expression, whence the latter becomes $\rho_0 S^2/c_0$. It is to be noted that this representation is possible whether the opening is in the form of a neck or is flat.

To get the expression for the dissipative force we need to calculate the amount of acoustical energy radiated from a hemispherical source of sound in a fluid. We shall give a rigorous derivation of this quantity in a later section (Sec. 3·2). It will suffice here to note that the final result for the dissipative force is

$$\frac{\rho_0 \omega k}{2\pi} S^2 \dot{\xi},$$

where $k = 2\pi/\lambda$ as usual. The reader should note the dependence on the velocity.

Finally, we must compute the stiffness coefficient. For this it is necessary to calculate the force acting on the area S of the opening. If the volume V of the resonator is decreased adiabatically by the amount dV, the excess pressure is (see eq. (1·14))

$$p' = \rho_0 c^2 s = -\rho_0 c^2 \frac{dV}{V},$$

for by definition, since the mass is constant, $d(V\rho) = 0$ and

$$s = \frac{\delta\rho}{\rho_0} = -\frac{dV}{V}.$$

Now $dV = -S\xi$, if the displacement producing the volume change is ξ. Therefore the force acting on area S is

$$\frac{\rho_0 c^2 S^2}{V} \xi.$$

We are now ready to write the equation of motion. Putting $S\xi = X$ and supposing that the resonator is driven by an external force producing a pressure p, we have

$$\frac{\rho_0}{c_0}\ddot{X} + \frac{\rho_0 \omega k}{2\pi}\dot{X} + \frac{\rho_0 c^2}{V}X = p. \tag{2·14}$$

Mathematically, if p is a harmonic function of the time, (2·14) is precisely similar to (2·1). For the steady state its solution therefore is

$$\dot{X} = \frac{p}{\frac{\rho_0 \omega k}{2\pi} + i\left(\frac{\rho_0 \omega}{c_0} - \frac{\rho_0 c^2}{V\omega}\right)}, \tag{2·15}$$

the real part of which written in trigonometric form becomes

$$\dot{X}_{\text{real}} = \frac{p_1 \cos(\omega t - \alpha)}{\rho_0 \sqrt{\left(\frac{\omega k}{2\pi}\right)^2 + \left(\frac{\omega}{c_0} - \frac{c^2}{V\omega}\right)^2}}, \tag{2·16}$$

where, as usual,

$$\tan \alpha = \frac{2\pi\left(\frac{\omega}{c_0} - \frac{c^2}{\omega V}\right)}{\omega k}. \tag{2·17}$$

The maximum value of $\dot{X}$ occurs approximately when $\omega/c_0 = c^2/V\omega$, that is, the approximate resonance value of ω is

$$\omega = \omega_0 = c\sqrt{\frac{c_0}{V}}. \tag{2·18}$$

The more accurate expression for the resonance frequency is

$$\omega_0 = c\sqrt{\frac{c_0}{V}} \cdot \frac{1}{\sqrt{1 + \frac{k^2 c_0^2}{8\pi^2}}},$$

but the term $k^2 c_0^2/8\pi^2$ is usually negligible, for $c_0 < < \lambda$, as a rule.

These theoretical conclusions are confirmed by experiments on resonators.

We now define *acoustic impedance* analogously to the mechanical impedance of the preceding section.

Thus, we write

$$Z = \frac{\text{pressure}}{\text{rate of volume displacement}} = \frac{\text{pressure}}{\text{volume current}}$$

$$= \frac{p}{\dot{X}}$$

$$= Z_1 + iZ_2,$$

where

$$Z_1 = \textit{acoustic resistance} = \frac{\rho_0 \omega k}{2\pi} = \frac{\rho_0 c k^2}{2\pi}, \tag{2·19}$$

and

$$Z_2 = \textit{acoustic reactance} = \rho_0 \left(\frac{\omega}{c_0} - \frac{c^2}{\omega V} \right). \tag{2·20}$$

It is customary to call the quantity ρ_0/c_0 the *inertance*, while the quantity $V/\rho_0 c^2$ is the *acoustic capacitance*. It is thus usual to write for the reactance $M\omega - 1/\omega C$ with $M = \rho_0/c_0$ and $C = V/\rho_0 c^2$. The latter is seen from eq. (2·14) to be the ratio of the volume displacement to the pressure for the case of static displacement. With regard to the former, it should be noted that the *inertance* is *not* the mass of the system. Rather we have the relation

$$\text{inertance} = \frac{\text{mass}}{S^2}.$$

As might be expected from the previous section the maximum $\dot{X}$ and the maximum pressure (or displacement) do not occur at quite the same frequency. The resonance frequency is, of course, given to a close approximation by (2·18), while the maximum displacement occurs for the value of ω which makes

$$\frac{\omega^4 k^2}{4\pi^2} + \left(\frac{\omega^2}{c_0} - \frac{c^2}{V} \right)^2$$

a minimum. This comes out to be

$$\omega_1 = c\sqrt{\frac{c_0}{V}} \cdot \frac{1}{\sqrt{1 + \dfrac{k^2 c_0^2}{4\pi^2}}}. \tag{2·21}$$

This is but slightly less than ω_0, as a rule.

The amplification constant of the resonator is the ratio of the squares of the maximum excess pressure in the resonator and the

maximum external operating pressure. For the former we have at once the expression,

$$p_{\max} = \rho_0 c^2 s_{\max} = \frac{\rho_0 c^2 X_{\max}}{V},$$

while the latter is simply p_1. Now we have

$$X_{\max} = \frac{p_1}{\rho_0 \omega \sqrt{\left(\frac{\omega k}{2\pi}\right)^2 + \left(\frac{\omega}{c_0} - \frac{c^2}{V\omega}\right)^2}}.$$

Hence after some reduction we arrive at

$$\textit{Amplification} = \frac{p^2{}_{\max}}{p_1{}^2} = \frac{1}{\left(\frac{k^3 V}{2\pi}\right)^2 + \left(1 - \frac{k^2 V}{c_0}\right)^2}, \qquad (2\cdot22)$$

which for the resonance case reduces simply to

$$\frac{4\pi^2}{k^6 V^2}.$$

2·4. Conductivity of an Orifice.—In Section 2·3 there was introduced the quantity c_0, which was called the acoustic conductivity of the opening into the resonator. This is an extremely important concept and will now be further examined. In the case of an opening in the form of a channel of length l it is possible, of course, to write down an approximate expression for the mass of the body of air that moves as a whole. The vibrations of this mass constitute one of the chief features of a resonator. But, as will later appear, the mass in the channel is not the entire effective mass of vibrating air and hence another quantity must be introduced on which the latter can be considered to depend. This quantity is the *conductivity*. Following Rayleigh [1] we write for the kinetic energy of a mass of fluid:

$$\text{K.E.} = \tfrac{1}{2}\rho_0 \int\int\int \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] dxdydz$$

$$= \tfrac{1}{2}\rho_0 \int\int\int (\nabla\varphi)^2 dxdydz \qquad (2\cdot23)$$

in the vector notation, where the integration is extended over the whole region occupied by the fluid whose motion has a sensible

[1] Rayleigh, Theory of Sound, Vol. II, Sec. 304.

value. By Green's theorem [1] this volume integral can be transformed into a surface integral over a surface including this whole volume. This theorem states that

$$\int\int\int[(\nabla\varphi)^2 + \varphi\nabla^2\varphi]dxdydz = \int\int\varphi\mathbf{n}\cdot\nabla\varphi dS, \qquad (2{\cdot}24)$$

where $\mathbf{n}\cdot\nabla\varphi$ is the component of $\nabla\varphi$ normal to the surface element dS. We can write $\mathbf{n}\cdot\nabla\varphi = \partial\varphi/\partial n$. Now the mass of fluid is supposed to move as a whole and hence $\dot{s}$ is approximately zero (where s is the condensation: see Sec. 1·12). Therefore in the moving fluid $\nabla^2\varphi$ is zero. The expression for the kinetic energy then reduces to

$$\text{K.E.} = \tfrac{1}{2}\rho_0\int\int\varphi\frac{\partial\varphi}{\partial n}dS. \qquad (2{\cdot}25)$$

Now on the surface over which the above integration is to be carried out, there will be no appreciable motion of the fluid and φ will therefore have a constant value. But it will obviously be different on the two sides of the orifice. Call the value on the inner side φ_1 and that on the outer side φ_2. Then if the flow is from left to right we have as the surface integral over S_1 and S_2:

Fig. 2·3.

$$\text{K.E.} = \tfrac{1}{2}\rho_0(\varphi_1 - \varphi_2)\int\int\frac{\partial\varphi}{\partial n}dS$$

$$= \tfrac{1}{2}\rho_0(\varphi_1 - \varphi_2)\dot{X}, \qquad (2{\cdot}26)$$

since the volume current $\dot{X} = \int\int(\partial\varphi/\partial n)dS$ by definition. To a first approximation $\dot{X}$ should be a linear function of $\varphi_1 - \varphi_2$. Hence we assume

$$\dot{X} = c_0(\varphi_1 - \varphi_2). \qquad (2{\cdot}27)$$

We note at once the analogy to Ohm's law in the case of electrical currents, whereby if a uniform difference of potential $E_1 - E_2$ is maintained between two points of a conductor, there flows a current $I = C(E_1 - E_2)$, where C is the reciprocal resistance or electrical

[1] See, for example, E. B. Wilson, Advanced Calculus, 1912, p. 349.

conductivity of the substance. One is therefore led to give the name acoustic conductivity to c_0. Writing the kinetic energy in terms of it, we obtain

$$\text{K.E.} = \tfrac{1}{2}\frac{\rho_0}{c_0}\dot{X}^2, \tag{2·28}$$

whence $\rho_0 S^2/c_0$ appears as the *effective* mass of the moving fluid in the opening.

We have already noted one application of the conductivity in the study of the Helmholtz resonator, where the potential and kinetic energies appear in the form:

$$\text{P.E.} = \tfrac{1}{2}\frac{\rho_0 c^2}{V}X^2 \tag{2·29}$$

$$\text{K.E.} = \tfrac{1}{2}\frac{\rho_0}{c_0}\dot{X}^2, \tag{2·30}$$

and the resonance frequency is given to a close approximation by

$$\omega_0 = c\sqrt{\frac{c_0}{V}}, \tag{2·31}$$

where V is the volume of the resonator chamber.

It should at once be emphasized that the computation of c_0 as defined in (2·27) is in general difficult and has been carried through mathematically in relatively few cases.

In the case of a simple aperture in an infinite plane wall of infinitesimal thickness the method pursued is to assume that φ is constant over the aperture, $\partial\varphi/\partial n$ is zero over the remainder of the plane wall and φ = constant at infinity (so that it may be put equal to zero). For the case of the ellipse this computation has been carried out [1] and for the special case in which the ellipse degenerates into a circle we have

$$c_0 = 2a, \tag{2·32}$$

where a is the radius.

Now no aperture is ever in a wall of infinitesimal thickness, and it is necessary to consider the influence of the thickness. Suppose the channel is of length l and of radius of cross section a. The conductivity of the channel alone would be simply $\pi a^2/l$, since the mass of the fluid in the channel is $\pi a^2 \rho_0 l$ and the effective mass by (2·28) is $\rho_0 \pi^2 a^4/c_0$. The effect of the orifice may be thought of as

[1] See Rayleigh, loc. cit., Sec. 306.

adding a conductivity in parallel (or resistance in series) with the channel. Hence as an approximation we write for the effective mass divided by S^2,

$$\rho_0/c_0 = (\rho_0 l/\pi a^2) + (\rho_0/2a),$$

or

$$c_0 = \frac{1}{\frac{l}{\pi a^2} + \frac{1}{2a}} = \frac{\pi a^2}{l + \frac{1}{2}\pi a}. \tag{2·33}$$

It is seen that we may consider the term $(\pi/2)a$ as a correction to the length of the channel. It is customary to write this in general in the form

$$c_0 = \frac{\pi a^2}{l + 2\alpha}, \tag{2·34}$$

where as a matter of fact $\alpha = (\pi/4)a$ is in general too small, though as l becomes very small in comparison with a, the assumed motion is approximately correct and α approaches $\frac{1}{4}\pi a$. It can be shown [1] that the above is an inferior limit and that the superior limit is $(8/3\pi)a$. Hence we must have the inequality expression for the actual value of α:

$$\frac{\pi a}{4} < \alpha < \frac{8a}{3\pi},$$

or $\alpha > .785a$ and $\alpha < .849a$, and the correct value of c_0 lies between the converging limits.

The connection between conductivity and the end correction of a cylindrical tube is further discussed from the standpoint of impedance theory in Section 6·6 and it is there shown that the correction to the end can be expressed in the form S/c_0, where S is the cross sectional area of the cylinder and c_0 is the conductivity of the *opening* of the tube. This should make clear the difference from (2·34), where c_0 refers to the conductivity of the whole channel.

2·5. Electrical Analogues.—The comparison of acoustical wave problems with electrical oscillations is often very valuable, but it must be made with caution and only with reference to the differential equation involved. The equation for the electrical oscillation problem analogous to the Helmholtz resonator is

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = Ee^{i\omega t}, \tag{2·35}$$

[1] See Rayleigh, loc. cit., Sec. 307.

where L is the inductance, R the resistance and C the capacitance of the circuit in which the oscillations take place. The impressed e.m.f. is $Ee^{i\omega t}$ and q is the charge, while the current is of course $\dot{q}$. The impedance is

$$\frac{Ee^{i\omega t}}{\dot{q}} = R + i\left(L\omega - \frac{1}{\omega C}\right). \tag{2·36}$$

The absolute value is the familiar expression

$$|Z| = \sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}$$

in terms of resistance and reactance. In examining the analogy with the acoustical case it is valuable to recall that in the electrical case we have to do with what may be called *distributed* impedance, while the acoustic impedance is a *point* impedance.

The phase angle is given by

$$\tan\theta = \frac{L\omega - \frac{1}{\omega C}}{R}. \tag{2·37}$$

The power factor, $\cos\theta = R/|Z|$, is an important quantity in the electrical case. Resonance occurs for the frequency for which

$$\omega = \frac{1}{\sqrt{LC}}. \tag{2·38}$$

The influence of the resistance on the resonance illustrates the damping effect discussed for the acoustical case in Section 2·2. In the electrical problem, however, it is generally customary to plot in the resonance curve the current against either the inductance or capacitance. For large and small resistances one then obtains curves similar in nature to Fig. 2·1 in Section 2·2.

It should be noted that the literature on modern acoustics shows wide divergences in the definition of acoustic impedance. For future convenience these are set down here. Kennelly and Crandall [1] define it as $\frac{\text{force}}{\text{particle velocity}}$, in analogy with mechanical impedance. A. G. Webster [2] used $\frac{\text{pressure}}{\text{volume displacement}}$, while Brillié

[1] Crandall, Theory of Vibrating Systems and Sound, 1926, p. 100.

[2] A. G. Webster, Proc. Nat. Academy Sci., 5, 275, 1919; H. Brillié, Le Génie Civil, 1919.

prefers $\frac{\text{pressure}}{\text{particle velocity}}$. This confusion is rather unfortunate and it might seem that the present work is adding to the confusion by adopting still another convention for this important quantity, namely $\frac{\text{pressure}}{\text{rate of volume displacement}}$ or $\frac{\text{pressure}}{\text{volume current}}$. The present authors believe, however, that this last definition has proved to be the most simple in the more recent work in acoustics and is now being widely used in technical literature.

2·6. Resonance of Tubes.—There are other common cases of resonance which provide some interest. These concern motion in a tube and are usually explained in elementary texts. It will be of value, however, to develop here the mathematical solution for the case of wave motion in a cylindrical tube, and then to point out the electrical analogy. Incidentally we shall obtain the resonance frequencies.

Let the tube be chosen with cross section S and let the distance of any point from one end be denoted by x. Assuming the possibility of motion in the x direction only and neglecting damping, the equation of motion (Sec. 1·13, eq. (1·17)) is

$$\ddot{\xi} = c^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{2·39}$$

wherein we use ξ, the displacement, instead of φ, the velocity potential. The solution in the case of a harmonic wave is, of course,

$$\xi = \xi_0 e^{ik(ct-x)} + \xi_1 e^{ik(ct+x)}, \tag{2·40}$$

where $k = \omega/c$ as usual and the two terms refer to waves going in the forward and backward directions through the tube, respectively. Now if the tube is *open* at both ends there will be (approximately at least) no change in pressure at the two ends.[1] From (1·14) and (1·26) it is seen that this condition means that $\partial\xi/\partial x = 0$ at the two ends, for $x = 0$ and $x = l$, if the length of the tube is l. Applying this condition to (2·40), it is found that we must have $\sin kl = 0$, whence $l = n\lambda/2$, where n is any integer. The resonance wave lengths are thus given by $\lambda = 2l/n$. On the other hand, if one end of the tube is closed, say at $x = 0$, and the other open, we must have $\xi = 0$ at $x = 0$, while $\partial\xi/\partial x = 0$ at $x = l$. This leads

[1] This assumption neglects the acoustic radiation at the openings, and hence the result will be approximate only. The magnitude of the error is estimated in Sec. 5·5.

to the result that in this case $l = (2n + 1)\lambda/4$, n being as before any integer, and the resonance wave lengths are $\lambda = 4l/(2n + 1)$. Similarly it can be shown that for a tube closed at both ends $\lambda = 2l/n$ as in the first case above. In the case of forced oscillations of a cylindrical tube it is of interest to note that the condition for resonance, if a prescribed harmonic source be maintained at one end, is $\lambda = 2l/n$ if the other end of the tube is *closed*, and $\lambda = 4l/(2n + 1)$ if the other end of the tube is *open*.

To understand the electrical analogy, let us calculate the *acoustic capacitance* of the tube. From Section 2·3 this is defined for the resonator as the ratio of volume displacement to the pressure. Since $\partial\xi/\partial x$ is the change in volume per unit volume, the capacitance is given by

$$\frac{Sl}{p} \cdot \frac{\partial \xi}{\partial x} = -\frac{Sl}{\rho_0 c^2}, \tag{2·41}$$

since $p = -\rho_0 c^2 \partial\xi/\partial x$. The acoustic capacitance per unit length of tube is then numerically $S/\rho_0 c^2$. Per unit cross section per unit length this reduces to $1/\rho_0 c^2$. Putting $S\xi = X$ and multiplying through by ρ_0, we can put eq. (2·39) into the form

$$\rho_0 \ddot{X} = \frac{1}{\frac{1}{\rho_0 c^2}} \frac{\partial^2 X}{\partial x^2}. \tag{2·42}$$

The corresponding electrical problem is that governed by the equation

$$L\ddot{q} = \frac{1}{C} \frac{\partial^2 q}{\partial x^2}, \tag{2·43}$$

which represents [1] the propagation of an electrical disturbance along a transmission line with distributed inductance and capacitance. The current is $I = \dot{q}$, the time derivative of the charge. The inductance per unit length and the capacitance per unit length are L and C respectively. In the transmission line the capacitance is between the line and its surroundings. The analogous acoustic capacitance, comparing (2·42) and (2·43), enters as if it were due to the elasticity of the tube, instead of the action of the fluid. If the fluid is confined in a tube whose walls are not rigid, the capacitance is thereby increased.

[1] See Fleming, Propagation of Electric Waves, 3d ed., 1919, p. 125.

2·7. Compound Resonators.—Let us now consider two Helmholtz resonators coupled as in Fig. 2·4, where the conductivities of the necks are c_{01} and c_{02} respectively, and V_1 and V_2 are the chamber volumes. The question at once arises, will the natural frequencies of the coupled resonators be the same as those of the two regarded separately? Rayleigh has shown [1] that this is not so.

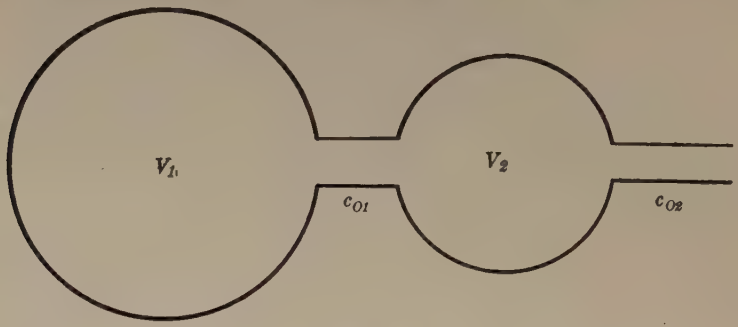

FIG. 2·4.

For the case where $V_1 = V_2 = V$ and $c_{01} = mc_{02}$ (m being any number), he has calculated the natural frequencies of the coupled system to be

$$\nu = \frac{\omega}{2\pi} = \frac{c}{2\pi}\sqrt{\frac{c_{02}}{2V}\left\{m + 2 \pm \sqrt{m^2 + 4}\right\}}. \qquad (2.44)$$

For the details of the calculation, which is based on the solution of the Lagrangian equations of motion for the system, reference should be made to Rayleigh's book. Moreover, the same author [2] has more recently extended the reasoning to show that when V_1 and V_2 are not equal, the ratio of the actual frequency ν of the compound resonator to that of either taken separately (assuming that the isolated frequencies ν_1 and ν_2 are equal) is given by

$$\frac{\nu^2}{\nu_1^2} = 1 + \frac{V_2}{2V_1} \pm \sqrt{\frac{V_2}{V_1} + \frac{V_2^2}{4V_1^2}}. \qquad (2·45)$$

An interesting special illustration of a compound resonator is afforded by the Boys type. This consists (Fig. 2·5) of a long open cylindrical tube coupled at one end to a Helmholtz resonator through

[1] See Rayleigh, Theory of Sound, 1896, Vol. II, Sec. 310.
[2] Rayleigh, Phil. Mag., 36, 231, 1918.

a neck of conductivity c_0. Paris[1] has shown that the natural frequencies ν of the combination are determined by the following equation:

$$\tan\frac{\pi\nu}{2\nu_1} = -\frac{2\pi\sigma\nu}{cc_0}\left(1 - \frac{\nu_2^2}{\nu^2}\right), \qquad (2\cdot46)$$

where ν_1 and ν_2 are the fundamental frequencies of the isolated tube and Helmholtz resonator respectively, σ is the cross sectional area of the tube and c is, as usual, the velocity of sound in air.

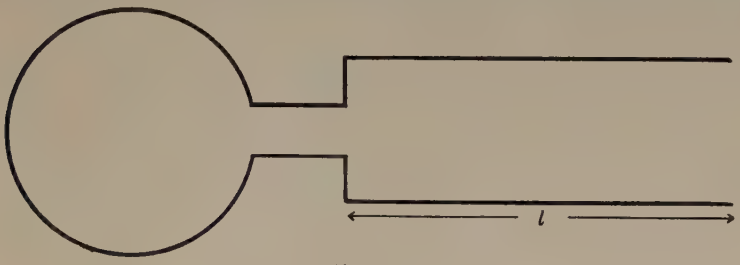

FIG. 2·5.

This type of resonator has proved of value in hot-wire microphone work (see Sec. 8·4). The formula (2·46) checks very well with experimental observations. Paris has extended his studies to include one or two Helmholtz resonators attached at the side of a cylindrical tube.[2] This work was done at the Acoustical Research Section of the Signals Experiment Establishment at Woolwich (England).

QUESTIONS AND PROBLEMS

1. In the case of the forced oscillations of the material system the differential equation of motion of which is

$$m\ddot{\xi} + K_1\dot{\xi} + f\xi = Sp_0e^{i\omega t},$$

show that ξ_{max} occurs for the frequency:

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{f}{m} - \frac{K_1^2}{2m^2}}.$$

For a system in which $m = 100$ gm., $K_1 = 200$ dynes sec/cm and $f = 1.6 \times 10^7$ dynes/cm, calculate ν_1 and ν_0 $(= 1/2\pi\sqrt{f/m})$, and calculate percentage difference.

[1] E. T. Paris, Proc. Roy. Soc. A, **101**, 393, 1922.
[2] E. T. Paris, Phil. Mag., **48**, 769, 1924.

2. Prove that the free oscillation frequency of the system in Problem 1, i.e., the frequency with which the system will oscillate if disturbed from equilibrium and allowed to vibrate freely, is

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{f}{m} - \frac{K_1^2}{4m^2}}\,.$$

In particular solve the differential equation to find ξ in this case and, using the data in Problem 1, calculate how long it will take the system to have its amplitude reduced to $1/e$ of its original value (i.e., the so-called decay modulus).

3. The number of periods it takes a given tuning fork of frequency 256 cycles to have its amplitude diminished in the ratio $1/e$ is about 5900. Find the value of the interval ω/ω_0, in which the intensity of resonance (measured by $\bar{W}$) falls to one-half its maximum value. Do the same for an air resonator for which the corresponding number of periods is 3300. Compare the results and comment on them. Plot curve for $\bar{W}$, using data of Problem 1 and taking Sp_0 arbitrary. Plot for two values of K_1, say first equal to 200 dynes sec/cm and second 400 dynes sec/cm.

4. In the case of a Helmholtz (or any) resonator, does the resonator create energy? Then why is there an increase in energy content in the space occupied by the resonator? All types of wind musical instruments utilize a resonating volume of air and evidently the amount of the acoustical output is markedly increased thereby. Reconcile this fact with your answer to the first question above.

5. Is a resonance receiving instrument essentially a space integrating or a time integrating instrument? Explain.

6. Judging from Section 2·7 on compound resonators, what general statement can be made concerning the retention of individual natural frequencies when two systems are (1) closely and (2) loosely coupled?

7. Given the following data for a Helmholtz resonator of frequency 256:

Diameter of sphere = 13 cm,
" " opening = 3.02 cm,
Volume of resonator chamber = 1053 cc.

Compute the conductivity c_0 of the opening and compare with the theoretical value. Compute the amplification constant of this resonator at its resonance frequency.

8. Deduce the exact values of the frequency which give X and $\dot{X}$ (for the Helmholtz resonator) maximum values.

9. Deduce the resonance wave lengths for a cylindrical tube of length l, when

(a) One end is open and the other closed.

(b) Both ends are closed.

Discuss the physical validity of the boundary conditions used.

10. Show that if a prescribed harmonic source of sound maintains a displacement $\xi = A \cos \omega t$ at one end of a cylindrical tube of length l, the condition for resonance (neglecting dissipation) is $\lambda = 2l/n$, if the other end of the tube is closed, and $\lambda = 4l/(2n + 1)$, if the other end is open (n being any integer). Discuss the same problem when damping is taken into account and show that the same result follows approximately.

11. Prove that the average rate of transfer of energy to a vibrating system by an external force is equal to the average rate of dissipation (see Sec. 2·2).

12. In eq. (2·14) the applied pressure is said to be the sum of three pressures. Do these three exist at the same point? Is there any assumption, not stated in the text, involved in this equation?

13. Justify the statement preceding eq. (2·27) to the effect that $\dot{X}$ is a linear function of $\varphi_1 - \varphi_2$.

14. Derive the differential equation for the free transverse vibrations of a uniform, flexible, stretched string. Integrate the equation for the case of simple harmonic vibrations subject to the boundary condition that the string is finite in length and fastened at each end. Determine the natural frequencies of vibration.

CHAPTER III

Transmission.—Changes in Area of Wave Front

3·1. Transmission in a Plane Wave.—It has already been pointed out (Sec. 1·15) that the propagation of an acoustic wave is accompanied by the transmission of energy through the medium. The power is the flow of energy per second. In Section 1·15 the calculation of power was made by finding the energy density and multiplying it by the velocity of sound, yielding the power transmission *per unit area* perpendicular to the direction of propagation. This quantity was then defined as the *intensity*. Another way of obtaining the power transmission will obviously be to calculate the rate at which work is done on the medium. But this, for any particular place and instant and for each unit area, is equal to the product of *excess pressure* and *particle velocity*.

The expressions for the pressure and the corresponding particle velocity in the case of a plane wave have already been given in eqs. (1·23) and (1·24). They are respectively

$$p = kc\rho_0 A \sin k(ct - x) \tag{3·1}$$

and

$$\dot{\xi} = kA \sin k(ct - x). \tag{3·2}$$

It is to be noted that the phase difference between these two quantities is zero. In the electrical analogue the power factor (the cosine of the phase difference between the electromotive force and the current—see Sec. 2·5) is thus unity. Consequently the average power transmission per unit area is simply

$$P = \frac{1}{2} k^2 A^2 c\rho_0, \tag{3·3}$$

if we recall that the time average of $\sin^2 k(ct - x) = 1/2$.[1] An examination of (3·1) in the light of the latter fact shows at once that

[1] The time average of any periodic function $f(x, t)$ as here understood is

$$\frac{1}{\tau}\int_0^\tau f(x, t)dt,$$

where the time interval τ is *large* compared with one period.

$$P = \frac{\overline{p^2}}{c\rho_0}, \tag{3·4}$$

the bar indicating, as usual, time average. Hence P, the average power transmission per unit area, is equal to I, the intensity as defined in Section 1·15.

The acoustic impedance in this case is given by

$$\frac{p}{\dot{X}} = \frac{p}{S\dot{\xi}} = \frac{\rho_0 c}{S}. \tag{3·5}$$

The impedance is thus a real quantity and represents a resistance. It is often denoted as the *radiation resistance* of the medium. It is worth noting that for a given medium the acoustic impedance of a progressive plane wave is everywhere the same provided the size of the wave front does not alter. This is the situation, for example, in a cylindrical tube with cross section small compared with the wave length and of infinite length (i.e., wave in one direction only). The quantity $\rho_0 c$, i.e., the quantity which when divided by S yields the acoustic or radiation resistance of the medium for a plane wave, will be called hereafter the *specific acoustic resistance* for a plane wave. It is the acoustic resistance *for* (not *per*) unit area, following the analogy of specific resistance in electrical theory. In general then we shall term $p/\dot{\xi}$ the specific acoustic impedance. For a plane wave it is, as we have just seen, a specific resistance. But this is not generally true, as we shall see in the next section in the discussion of spherical waves.

3·2. Transmission in a Spherical Wave.—The transmission in a spherical wave differs in certain important respects from that in a plane wave. From Section 1·14 (eqs. (1·37) and (1·41)) we have for the excess pressure and corresponding particle velocity in a diverging spherical wave the respective expressions,

$$p = -ickA\rho_0/r \cdot e^{ik(ct-r)}$$

and

$$\dot{\xi} = -(1/r + ik)A/r \cdot e^{ik(ct-r)}.$$

Let us first consider the impedance. The quantity $p/\dot{\xi}$ has been defined as the specific acoustic impedance. From the above we then have

$$\frac{p}{\dot{\xi}} = \frac{i\rho_0 ck}{\left(\frac{1}{r} + ik\right)} = \rho_0 ck\left(\frac{kr^2}{1 + k^2r^2} + i\frac{r}{1 + k^2r^2}\right). \tag{3·6}$$

Another way of writing this quantity is

$$\frac{p}{\dot{\xi}} = \frac{\rho_0 kc}{\sqrt{\frac{1}{r^2} + k^2}} e^{i\theta}, \tag{3·7}$$

where θ is the phase angle between p and $\dot{\xi}$, and is of such magnitude that

$$\tan\theta = \frac{1}{kr}. \tag{3·8}$$

It is generally desirable to split the specific impedance into its real and imaginary parts. Thus we write

$$Z_s = Z_{s1} + iZ_{s2}, \tag{3·9}$$

where

$$Z_{s1} = \rho_0 ck \frac{kr^2}{1 + k^2r^2} \tag{3·10}$$

and

$$Z_{s2} = \rho_0 ck \frac{r}{1 + k^2r^2}; \tag{3·11}$$

Obviously

$$|Z_s| = \frac{\rho_0 ckr}{\sqrt{1 + k^2r^2}}. \tag{3·12}$$

Z_{s1} is the specific acoustic *resistance*, while Z_{s2} is the specific acoustic *reactance;* the latter in this case, since it is positive, is a specific inertance, following the definition given in Section 2·3.

The *total* acoustic impedance is given by

$$Z = \frac{p}{\dot{X}} = \frac{p}{4\pi r^2 \dot{\xi}}, \tag{3·13}$$

which in absolute value is

$$|Z| = \frac{\rho_0 ck}{4\pi r\sqrt{1 + k^2r^2}}. \tag{3·14}$$

So far we have considered a diverging wave only. We next investigate the case of the converging wave, for which the velocity potential is

$$\varphi = \frac{A}{r} e^{ik(ct+r)}. \tag{3·15}$$

Then

$$\dot{\xi} = \left(-\frac{1}{r} + ik\right)\frac{A}{r} e^{ik(ct+r)} \tag{3·16}$$

and

$$p = \frac{ickA\rho_0}{r} e^{ik(ct+r)}. \tag{3·17}$$

Consequently in this case we have for the impedance components, recalling that $\dot{\xi}$ is to be measured in the direction of decreasing r, i.e., the direction in which the wave is moving,

$$Z_{s1} = \rho_0 ck \frac{kr^2}{1 + k^2r^2} \tag{3·18}$$

and

$$Z_{s2} = -\rho_0 ck \frac{r}{1 + k^2r^2}. \tag{3·19}$$

The latter is the negative of (3·11). In other words, Z_{s2} is now the reciprocal of a *specific acoustic capacitance* instead of being a *specific inertance.*

The power factor is $\cos\theta$ and its value is

$$\cos\theta = \left(1 + \frac{1}{k^2r^2}\right)^{-1/2} = \frac{kr}{\sqrt{1 + k^2r^2}}. \tag{3·20}$$

From the electrical analogy it is seen that the average power transmission per unit area is given by

$$P = \frac{1}{2} p_{\max} \dot{\xi}_{\max} \cos\theta, \tag{3·21}$$

where $p_{\max}$ and $\dot{\xi}_{\max}$ are the maximum *real* values of pressure and velocity, respectively. But we have

$$p_{\max} = \frac{ckA\rho_0}{r}, \tag{3·22}$$

and

$$\dot{\xi}_{\max} = \frac{Ak}{r\cos\theta}. \tag{3·23}$$

Hence there finally results

$$P = \frac{ck^2A^2\rho_0}{2r^2}. \tag{3·24}$$

It is to be understood, of course, that (3·24) can be obtained directly by calculating the time average of the product of the real parts of p and $\dot{\xi}$. Indeed this is the way the formula (3·21) is derived.

The expression for P in (3·24) thus represents the intensity and shows that for a spherical wave this quantity varies inversely as

the square of the distance from the source. This important fact has already been used in an earlier portion of the book. Incidentally it is seen that the eq. (3·4) derived in Section 3·1 for plane waves also holds for spherical wave transmission (note that for p in the formula we must, of course, use the *real* part).

The expression for the specific acoustic resistance Z_{s1} given in (3·10) is particularly interesting and important as it enables us to calculate the dissipative force due to the radiation of sound energy from a source into a fluid medium. To be specific, suppose we have a small spherical source of sound of radius r_0. Then if r_0 is sufficiently small in comparison with the wave length the specific acoustic resistance at r_0 becomes [1]

$$Z_{s1} = \rho_0 c k^2 r_0^2. \tag{3·25}$$

Now if the sound is being radiated through a hemisphere of surface $S = 2\pi r_0^2$, we can write for the total acoustic resistance

$$Z_1 = \frac{\rho_0 c k^2}{2\pi} = \frac{\rho_0 \omega k}{2\pi}, \tag{3·26}$$

whence the dissipative force for the whole area, which is given by $Z_1 \dot{\xi} S^2$, is equal to

$$\frac{\rho_0 \omega k S^2}{2\pi} \dot{\xi}, \tag{3·27}$$

which is the expression which has already been used in the case of the opening to the Helmholtz resonator (Sec. 2·3). Strictly speaking, the above derivation applies only to the case of a vibrating sphere, but if the opening to the resonator is small compared with the wave length the comparison is allowable as an approximation.

The physical significance of the distinction between the acoustic impedance of a plane wave and that of a spherical wave is well illustrated in the following example. Let a plane wave be incident on an infinite plane wall with a small aperture. From this aperture there emerges an approximately hemispherical wave. The specific acoustic impedance of the incident wave is *real*, and p and $\dot{\xi}$ are in phase. The specific acoustic impedance of the emergent wave is *complex* and there is a phase difference between p and $\dot{\xi}$ (eq. (3·8)) which is practically 90° in the neighborhood of the aperture. In

[1] It can be shown (see Sec. 10·5) that the impedance expressions in the case of a finite source emitting spherical waves are, at any distance from the center of the source greater than the radius, the same as for a similar point source located at the center.

other words, a phase change of a quarter wave length is introduced by the transmission through the aperture and this may be attributed to the physical change in impedance.[1]

3·3. Absorption by the Medium.—Our discussion in the preceding section has indicated that when sound in a spherical wave is transmitted through a medium the power transmission should fall off as the inverse square of the distance from the source. As a matter of fact, in practice this is never quite true. In other words, in addition to the normal scattering due to spreading there is always present absorption of energy by the medium. For the causes of the absorption we may look to the viscosity of the medium, heat conduction, and in large scale transmission non-homogeneities in the structure of the medium, such as are produced by the effects of wind and temperature or by variation in density due to change in composition, etc. The effect of viscosity will now be considered. As might be expected this is not so important in the case of spherical waves as in that of plane waves in a tube.

We can best get at the matter by referring back to eq. (2·39) in Section 2·6, which is the equation of plane wave propagation in a tube, neglecting damping or frictional resistance. Introducing a dissipation force the more general equation of motion is

$$\rho_0 \ddot{\xi} + R' \dot{\xi} = \rho_0 c^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{3·28}$$

where R' is the damping coefficient per unit area. Careful analysis of the vibrations of a viscous fluid [2] shows that (3·28) will be the wave equation in such a fluid if we have

$$R' = \frac{4}{3} \mu k^2, \tag{3·29}$$

where $k = \omega/c$, as usual, and μ is the coefficient of viscosity. The solution to (3·28) is found in the usual way by assuming $\xi = \xi_0 e^{ik(ct-mx)}$, substituting into the equation and thereby determining m. The result proves to be, neglecting squares of small quantities,

$$\xi = \xi_0 e^{-(2/3)(\mu k^2/\rho_0 c)x} e^{ik(ct-x)}, \tag{3·30}$$

which represents a progressive wave with exponentially decreasing amplitude. It will be noted that the so-called *attenuation factor*,

[1] Attention is called to the analogous case in optics which is not so easily sensed.

[2] See H. Lamb, Dynamical Theory of Sound, 1925, p. 186.

$\frac{2}{3}(\mu k^2/\rho_0 c)$, depends on the square of the frequency. The actual damping for waves in air is very small for moderate frequencies (thus $\frac{2}{3}(\mu k^2/\rho_0 c) = .24 \times 10^{-8}$ for $\omega = 1000$ in air), but rises rapidly with increasing frequency. The influence on large scale transmission through air is believed to have been observed, but is rather hard to estimate exactly because of the effects of wind and temperature already emphasized (Secs. 1·7 and 1·8).

The effect of heat conduction, which is comparable with but certainly not greater than that of viscosity, need not detain us here. Reference may be made to the discussion in Lamb.[1] Viscosity may be expected to exert a greater influence when sound waves pass through a relatively narrow tube in which the wall offers great resistance to the motion of the air in contact with it. The solution in this case was first given by Helmholtz and complete details may be found in Rayleigh,[2] Lamb[3] or Crandall.[4] We are interested here only in the final result, which can be expressed in the same form as (3·30), namely

$$\xi = \xi_0 e^{-\alpha x} e^{ik'(c't-x)}, \tag{3·31}$$

where now we have

$$\alpha = \frac{1}{ac}\sqrt{\frac{\omega\mu}{2\rho_0}} \tag{3·32}$$

and

$$k' = \omega/c'.$$

In the above equations a represents the radius of the tube and c' is the modified phase velocity of the wave, being given by

$$c' = c\left(1 - \frac{1}{a}\sqrt{\frac{\mu}{2\omega\rho_0}}\right). \tag{3·33}$$

The theory underlying the development of these equations is only approximate and empirical formulas arising from experiments are doubtless more accurate. Recently[5] Wold and Stibitz have given the empirical formula,

$$c' = c\left(1 - \frac{2.23}{a(\omega/2\pi)^{.53}}\right),$$

which agrees fairly well with the form of (3·33). It is, at any rate,

[1] H. Lamb, loc. cit., p. 190.
[2] Rayleigh, loc. cit., I, p. 318.
[3] Lamb, loc. cit., p. 193.
[4] Crandall, loc. cit., p. 237.
[5] Science, LXVI, No. 1712, 1927.

interesting to note that the effect of the viscous damping in this case is to lower the velocity of the wave.

For tubes so narrow that the inertia and kinetic reaction of the fluid may be neglected in comparison with the frictional force, and where the condensations and rarefactions of the fluid are practically isothermal on account of the almost perfect heat conduction, Rayleigh [1] has shown that the solution may still be put into the general form (3·31); or more specifically we have

$$\bar{\xi} = Ce^{-\sqrt{(\omega R''/2p_0)}x} e^{i(\omega t - \sqrt{(\omega R''/2p_0)}x)}, \qquad (3\cdot34)$$

which gives the average displacement velocity instead of the displacement itself. The damping factor R'' in this case is

$$R'' = \frac{8\mu}{a^2}, \qquad (3\cdot34a)$$

which is the well known Poiseuille coefficient, commonly associated with the efflux of a fluid from a capillary tube. The quantity p_0 in the formula is the mean equilibrium value of the pressure in the tube. Rayleigh and others have made important application of (3·34) to the deadening sound in the interstices in carpets and curtains.

The damping due to absorption in transmission of sound waves in three dimensions through water is not primarily due to viscosity. In fact, it is decrease in intensity produced by scattering, which in turn is caused by the non-homogeneities in the structure of the water. It is interesting to note that, at any rate, the absorption in this case *is* still an exponential function of the distance from the source and hence causes a more rapid decrease of intensity than would occur with an inverse square function. This matter is of great importance in sound signalling in water and will be further discussed in the chapter on that subject.

3·4. Sound Transmission in Pipes.—Constantinesco System of Hydraulic Power Transmission.—We can utilize the material of the preceding section to discuss briefly the hydraulic transmission of energy through a pipe line, first developed by M. Constantinesco [2] during the late war. If we have a rigid pipe and vibrations are communicated to the enclosed fluid (generally water) by the alter-

[1] Rayleigh, loc. cit., p. 327. See also Lamb, loc. cit., p. 197.

[2] M. Constantinesco, Theory of Sonics, London, 1920.

nating motion of a piston in one end, energy will be transmitted to a piston placed at the other end. In the practical application the frequency is of the order of 100 cycles/sec.

If we neglect the wave reflected from the output end, the theory of the transmission becomes fairly simple. The displacement at any point distant x from the input end is (see eq. (3·31) in Sec. 3·3)

$$\xi = \xi_0 e^{-\alpha x} e^{i(\omega t - \beta x)}, \tag{3·35}$$

where for the moment we leave aside the exact evaluation of α and β. The velocity and excess pressure are given by

$$\dot{\xi} = i\omega_0 e^{-\alpha x} e^{i(\omega t - \beta x)}, \tag{3·36}$$

$$p = -\rho_0 c^2 \frac{\partial \xi}{\partial x} = \rho_0 c^2 (\alpha + i\beta) \xi_0 e^{-\alpha x} e^{i(\omega t - \beta x)}. \tag{3·37}$$

To get the power transmission, we must find the time average of the product of $\dot{\xi}_{\text{real}} \cdot p_{\text{real}}$. Now

$$\dot{\xi}_{\text{real}} = -\omega \xi_0 e^{-\alpha x} \sin(\omega t - \beta x),$$
$$p_{\text{real}} = \rho_0 c^2 \xi_0 e^{-\alpha x} [\alpha \cos(\omega t - \beta x) - \beta \sin(\omega t - \beta x)].$$

Hence

$$P = \overline{\dot{\xi}_{\text{real}} p_{\text{real}}} = \frac{1}{2} \omega \rho_0 c^2 \beta \xi_0^2 e^{-2\alpha x}. \tag{3·38}$$

If the length of the line is l, the efficiency or power transmission ratio is therefore

$$P_r = e^{-2\alpha l}. \tag{3·39}$$

More extended analysis [1] which takes into account the return wave from the output end indicates that for *maximum* power transmission we should have

$$\sin \beta l = 0$$

or

$$l = n\pi/\beta, \tag{3·40}$$

where n is any integer. It is of interest to note that this is likewise the condition for maximum response in the case of forced oscillations of a tube closed at one end. (See Prob. 10, Chap. II.)

Concerning the exact values of α and β there seems to be some confusion. The general custom seems to be to use the Poiseuille coefficient in computing α. As the discussion in the Section 3·3 indicates, however, this procedure is valid only if the pipe line is so narrow that the kinetic reaction of the fluid can be neglected in

[1] See Crandall, loc. cit., p. 101.

comparison with the frictional forces. For the larger tubes actually in use, it would appear more satisfactory, therefore, to use the values indicated in (3·32) of Section 3·3.

It should be emphasized that the above considerations are based on the assumption that the pipe is rigid. A more careful analysis would show that the elasticity of the pipe must be taken into consideration. Its effect naturally will be to increase α and reduce the efficiency and the velocity materially. For further details concerning the practical application of the system, which up to the present time has been rather limited, reference should be made to Drysdale et al., "The Mechanical Properties of Fluids," 1924, pages 221 ff. and 321, and also to Constantinesco's book above mentioned.

Somewhat allied to the transmission problem above considered is the Kundt's tube experiment in which forced oscillations in a tube closed at one end are produced by a piston at the other end. The resonance condition for this case has already been mentioned in Section 2·6 and is the same as that for maximum transmission in the Constantinesco system, though the phenomena are physically somewhat different in the two cases. (See Crandall, loc. cit., for the details.) Most standard texts discuss the Kundt's tube thoroughly, but there is one point of interest which may be worth mentioning here. It is usually stated that the piston end is approximately but not exactly at a *node*, implying that there is an exact node in the near vicinity. One of the authors [1] has shown that this view is incorrect. It is more correct to say that the piston end is *at an approximate node*, that is, this end has its mean position at a point which approximates a node more nearly than any other point along the tube in the vicinity.

In the passage of sound through long tubes of fairly large diameter (i.e., voice tubes of diameter one inch to four inches) there is an attenuation which is not primarily the result of viscosity or of heat conduction but is probably due to absorption by the walls of the tube.[2] Reference may here be made to experiments recently performed on voice tubes by Eckhardt and others.[3] Long tubes such as are used on shipboard were studied, assuming exponential

[1] G. W. Stewart, . . . (not yet published).

[2] The investigators here mentioned favor the interpretation that the attenuation is due particularly to "skin friction."

[3] E. A. Eckhardt, V. L. Chrisler, P. P. Quayle, M. J. Evans and E. Buckingham, Technologic Papers of the Bureau of Standards, 21, 163 (1926–27).

absorption of sound on passage through them. The measurements give the values of an attenuation coefficient α defined as follows

$$\alpha = 2.303/l \cdot \log_{10} I_0/I,$$

where I_0 is the initial intensity and I that after passing through a length l feet of tubing. The tubes used were made of sections 10 feet in length. A few sample results will be noted here. For straight tubing of brass one inch in diameter the mean value of α over a range of frequencies from 254 to 3280 is 0.041, corresponding to a reduction in intensity to 66% of its original value after passage through 10 feet of tubing. Increase of the diameter to 4 inches decreases α to 0.015, corresponding to a reduction in intensity to 86% for a 10-foot section. For fiber tubes the corresponding values of α are about 50% greater than those for brass, a result understandable on the basis of the assumption of absorption by the walls. Sweeps consisting of bends of 90° and 10 feet in over all length were also investigated. Thus for a 2 inch brass bend the average α comes out to be 0.030, decreasing to 0.018 for a 4 inch bend. Flexible tubes show much higher attenuation. The article referred to contains the results of measurements on the effects of various fittings, including cone terminals and inserted diaphragms. Many of the values given are unfortunately probably in considerable error due to the difficulty of avoiding resonance effects.

The same investigators also report measurements on the articulation (see Sec. 11·4) in sound transmitted through voice tubes.

3·5. Transmission in Pipes.—Change of Area at a Junction.— Consider two pipes joined as in the accompanying figure (Fig. 3·1) with sound traveling from the pipe of cross section S_1 to that of cross section S_2. The problem is to find the power transmission through the junction, assuming that there is no return wave from the conduit at the right. The result will then apply strictly to an infinite tube in which there is no damping. Nevertheless in practice the return wave will often be safely negligible. For the displacements in the incident, reflected and transmitted waves at the junction (which we shall take to correspond to $x = 0$) we shall write:[1]

$$\xi_i = A_0 e^{i\omega t}, \qquad \xi_r = A_1 e^{i\omega t}, \qquad \xi_t = A_2 e^{i\omega t}, \tag{3·41}$$

wherein the amplitudes A_1 and A_2 are assumed complex in order to

[1] It should be understood here and in all cases of acoustic transmission that unless otherwise specified the existence of a steady state is implied.

take care of general phase differences. From eqs. (1·14) and (1·26) and the discussion accompanying the latter, we may write

$$p = \mp \rho_0 c^2 \frac{\partial \xi}{\partial x},$$

where the minus or plus sign is to be taken according as the wave is progressing in the *positive* or *negative* direction. We then have

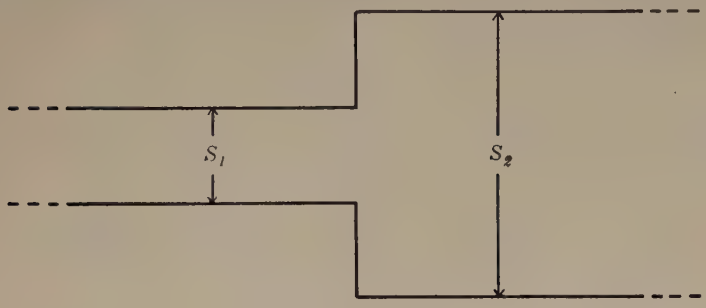

FIG. 3·1.

(applying the above equation to ξ in the form given by eq. (1·51), for example) for the excess pressures at the junction the corresponding expressions:

$$\begin{aligned} p_i &= i\omega R_s A_0 e^{i\omega t}, \\ p_r &= i\omega R_s A_1 e^{i\omega t}, \\ p_t &= i\omega R_s A_2 e^{i\omega t}, \end{aligned} \tag{3·42}$$

wherein we denote $\rho_0 c$ by R_s, for it is by eq. (3·5) the specific acoustic resistance of a plane wave. It should also be recalled that ξ_r is measured in the negative x direction. The boundary conditions to be satisfied are (see Sec. 1·18):

(1) continuity of pressure,

(2) continuity of discharge rate or volume displacement. That is, we must have at $x = 0$

$$p_i + p_r = p_t \tag{3·43}$$

and

$$S_1(\xi_i - \xi_r) = S_2 \xi_t, \tag{3·44}$$

where the minus sign in eq. (3·44) recalls that ξ_r is measured in the negative x direction. These two conditions yield respectively

$$A_0 + A_1 = A_2 \tag{3·45}$$

and

$$S_1(A_0 - A_1) = S_2 A_2. \tag{3·46}$$

If we call $S_2/S_1 = m$, we have therefore

$$\frac{A_0 - A_1}{A_0 + A_1} = m,$$

or, solving for A_1 and A_2 in terms of A_0,

$$A_1 = -A_0\left(\frac{m-1}{m+1}\right) \tag{3·47}$$

and

$$A_2 = A_0\left(\frac{2}{m+1}\right). \tag{3·48}$$

From the above equations we draw at once the following conclusions:

(1) If A_0 is real, then both A_1 and A_2 must be real.

(2) The transmitted wave is always in the same phase as the incident wave with respect to pressure and displacement (from eq. (3·48)).

(3) If $m < 1$, A_1 and A_0 have the same sign and hence the incident and reflected waves are in the *same* phase for both displacement and pressure. But if $m > 1$, the incident and reflected waves are *opposite* in phase. We note that, as in Section 1·18, the change in phase is either zero ($m < 1$) or π ($m > 1$). If $m = 1$, there is, of course, no reflected wave at all.

The average flow of energy in the incident wave is given by $\overline{S_1 p_i \dot{\xi}_i}$, while the transmitted flow is $\overline{S_2 p_t \dot{\xi}_t}$. We have therefore for the power transmission ratio:

$$P_r = \frac{\text{Transmitted flow}}{\text{Incident flow}} = \frac{\overline{S_2 p_t \dot{\xi}_t}}{\overline{S_1 p_i \dot{\xi}_i}} = \frac{S_2 A_2^2}{S_1 A_0^2} = \frac{4m}{(m+1)^2}$$

$$= 1 - \left(\frac{m-1}{m+1}\right)^2. \tag{3·49}$$

It thus appears that the percentage of energy reflected is

$$\left(\frac{m-1}{m+1}\right)^2 = \left(\frac{S_2 - S_1}{S_2 + S_1}\right)^2. \tag{3·50}$$

This is the result already stated in Section 1·5.

The reader will note that the effect on the phase of a plane wave in a conduit when reflection occurs as a result of transition to a *smaller* cross section is analogous to that which we have seen to be associated with reflection of sound going from a *rare* to a

dense medium. Likewise the transition from *small* cross section to *large* is analogous to the passage of sound from a *dense* to a *rare* medium. We shall have occasion to make use of this analogy in our further study of transmission problems.

3·6. Change of Area at Two Junctions.—Constriction and Expansion.[1]—Figure 3·2 represents two conduits connected by a third

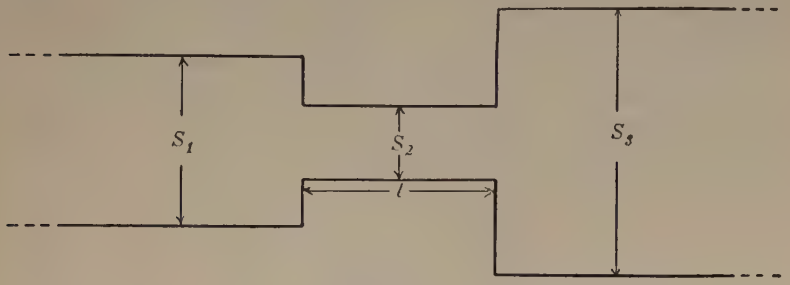

FIG. 3·2.

conduit or channel of length l. The cross sectional areas are S_1, S_2, and S_3 respectively. The boundary between S_1 and S_2 is denoted by I and that between S_2 and S_3 by II. Sound energy is supposed to pass through the system from left to right and it is desired to find the yield, or power transmission ratio, assuming no absorption damping. At I we expect to have an incident and reflected wave in S_1 and an incident and reflected wave in S_2. At II there will likewise be an incident and reflected wave in S_2 and a transmitted wave in S_3, neglecting as usual the return wave in S_3. The incident and reflected wave displacements in S_1 will be denoted by

$$A_1 e^{i\omega t} \quad \text{and} \quad B_1 e^{i\omega t}, \tag{3·51}$$

respectively. The displacements of waves of corresponding directions (at I) in S_2 are

$$A_2 e^{i\omega t}, \quad B_2 e^{i\omega t}, \tag{3·52}$$

wherein we have assumed the coefficients to be complex to take care of all phase differences. If (3·52) gives the two displacements

[1] Much of the material in this and the following sections of this chapter follows with certain modifications the work of H. Brillié, Le Genie Civil, **75**, 223 (1919).

at I in S_2, the corresponding two components in S_2 at II are

$$\begin{matrix} A_2 e^{-ikl} e^{i\omega t}, \\ B_2 e^{ikl} e^{i\omega t}. \end{matrix} \tag{3·53}$$

Lastly we may write, for the transmitted wave displacement at II in S_3,

$$A_3 e^{i\omega t}.$$

Again x need not appear, for A_3 is complex and contains the phase.

At each surface we have the two necessary conditions:

(1) continuity of pressure,

(2) continuity of volume displacement or of volume current.

On substitution we have therefore the corresponding equations for boundary I:

$$\begin{aligned} A_1 + B_1 &= A_2 + B_2, \\ S_1(A_1 - B_1) &= S_2(A_2 - B_2). \end{aligned} \tag{3·54}$$

In the above it should be noted that the minus signs in the displacement condition are due to the fact that the displacements $B_1 e^{i\omega t}$ and $B_2 e^{i\omega t}$ are, because of the definition of positive displacement in this text, measured in the negative x direction. At the boundary II we have

$$\begin{aligned} A_2 e^{-ikl} + B_2 e^{ikl} &= A_3, \\ S_2(A_2 e^{-ikl} - B_2 e^{ikl}) &= S_3 A_3. \end{aligned} \tag{3·55}$$

Calling $S_2/S_1 = m_1$ and $S_3/S_2 = m_2$, it follows from eq. (3·54) that

$$\begin{aligned} A_1 &= \tfrac{1}{2}[(m_1 + 1)A_2 - (m_1 - 1)B_2], \\ B_1 &= \tfrac{1}{2}[(m_1 + 1)B_2 - (m_1 - 1)A_2]. \end{aligned} \tag{3·56}$$

Similarly from the second set of conditions it follows that

$$\begin{aligned} A_2 e^{-ikl} &= \tfrac{1}{2}(m_2 + 1)A_3, \\ B_2 e^{ikl} &= -\tfrac{1}{2}(m_2 - 1)A_3, \end{aligned} \tag{3·57}$$

from which, by substitution into eq. (3·56) above, we finally have on reduction

$$A_1 = \frac{A_3}{2}[(m_1 m_2 + 1)\cos kl + i(m_1 + m_2)\sin kl]. \tag{3·58}$$

Similarly,

$$B_1 = -\frac{A_3}{2}[(m_1m_2 - 1)\cos kl + i(m_1 - m_2)\sin kl]. \quad (3{\cdot}59)$$

Now we have seen that the power transmission ratio is the ratio of the average energy flow in the transmitted wave in S_3 to the average energy flow in the incident wave in S_1. It is then $P_r = S_3\overline{p_t\dot{\xi}_t}/S_1\overline{p_i\dot{\xi}_i}$. The only question arises as to the treatment of the complex amplitudes. This is easily handled, but the discussion is given here in full for future reference. Let $A_1 = a_1 + ib_1$, say. Then we have $\xi_i = A_1e^{i\omega t}$ and, since $p_i = \rho_0 c\dot{\xi}_i$ (eq. (1·45)),

$$\dot{\xi}_{t,\,\text{real}} = -a_1\omega\sin\omega t - b_1\omega\cos\omega t,$$

$$p_{t,\,\text{real}} = -\rho_0 c[a_1\omega\sin\omega t + b_1\omega\cos\omega t].$$

Therefore

$$\overline{\dot{\xi}_{t,\,\text{real}}\cdot p_{t,\,\text{real}}} = \frac{\rho_0 c\omega^2}{2}(a_1^2 + b_1^2) = |A_1|^2\frac{\rho_0 c\omega^2}{2}. \quad (3{\cdot}60')$$

It therefore follows at once that

$$P_r = \frac{S_3|A_3|^2}{S_1|A_1|^2} = m_1m_2\frac{|A_3|^2}{|A_1|^2}. \quad (3{\cdot}60)$$

But from eq. (3·58) above, we have

$$|A_1|^2 = \tfrac{1}{4}|A_3|^2[(m_1m_2 + 1)^2\cos^2 kl + (m_1 + m_2)^2\sin^2 kl]$$

and therefore the transmission ratio becomes

$$P_r = \frac{4m_1m_2}{(m_1m_2 + 1)^2\cos^2 kl + (m_1 + m_2)^2\sin^2 kl} \quad (3{\cdot}61)$$

$$= \frac{4m_1m_2}{(m_1m_2 + 1)^2}\cdot\frac{1}{1 - \dfrac{(m_1^2 - 1)(m_2^2 - 1)}{(m_1m_2 + 1)^2}\sin^2 kl}.$$

It will now be observed from the above:

(1) When kl is very small, the transmission is independent of the cross sectional area of the channel or mid section.

(2) If $\sin kl = \pm 1$, we have for the power transmission

$$P_r = \frac{4m_1m_2}{(m_1 + m_2)^2}, \quad (3{\cdot}62)$$

which, it is interesting to notice, becomes equal to unity if $m_1 = m_2$ or $S_2^2 = S_1 S_3$, i.e., if the cross section of the channel is the geometrical mean of those of the two conduits.

(3) If $\sin kl = 0$, we have

$$P_r = \frac{4m_1m_2}{(m_1m_2 + 1)^2}, \tag{3·63}$$

which equals unity for $m_1m_2 = 1$ or $S_1 = S_3$. Under these conditions the introduction of the constriction or channel does not alter the transmission, except of course through absorption, which we are neglecting in the present discussion. Now since $kl = 2\pi l/\lambda$, it follows that for a given length of channel the particular wave lengths $\lambda = 2l/n$ (where n is any integer) are transmitted most intensely. In particular we get complete transmission for these wave lengths if $S_3 = S_1$ independently of the cross section of the channel.

In the case where $m_1 = m_2$, we get complete transmission or unit yield for the wave lengths $\lambda = 4l/(2n + 1)$, where n is any integer. In general, then, the interposition of a channel either constricting or expanding in nature renders the conduit selective.

That there is a difference in phase between the incident wave displacement and that of the finally transmitted wave is seen at once from eq. (3·58). We can get its magnitude by writing

$$A_1 = \frac{A_3}{2} e^{i\theta},$$

whence we have at once

$$\tan\theta = \frac{m_1 + m_2}{m_1m_2 + 1} \tan kl. \tag{3·64}$$

This shows that A_1 and A_3 do not differ in phase by kl as one might easily suppose, and that this actual phase difference depends upon m_1 and m_2 as well as upon kl. The practical point to be observed is that, if one wishes to keep the phase of the transmitted wave constant, one must not make intensity adjustments by pinching or constricting the tube. The above ideal case shows that a small constriction cannot change intensity anyway, and change of phase is often a matter of importance in acoustic experiments. It should again be emphasized that we are taking no account of viscous damping in the above discussion. However, the treatment of

Section 3·3 assures us that this effect will be small unless the cross sections of tube and channel are very small.

Experiments have been carried out to test the formula (3·61) both for constrictive and expansive channels. The results, which are in good accord with the theory, are referred to more specifically in Section 7·1 in connection with simple methods of sound filtration.

3·7. Conical Pipes.—We first consider briefly the resonance properties of a cone. This will then lead up to the use of conical pipes for transmitting sound.

The accompanying figure (Fig. 3·3) represents a frustrum of a right circular cone with vertex at O. Let the distance from O of

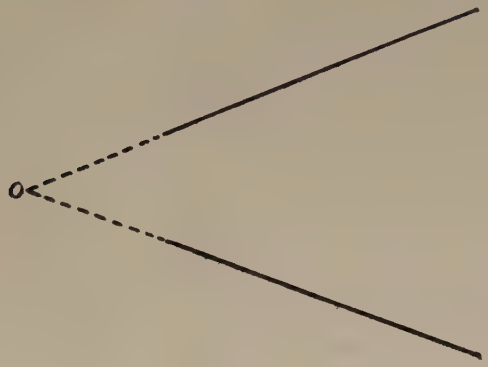

FIG. 3·3.

any point on the bounding circle at the smaller end be r_1 and the corresponding distance for the larger end be r_2. By virtue of symmetry the velocity potential at any point distant r from O is (see Sec. 1·14)

$$\varphi = \frac{A}{r} e^{ik(ct-r)} + \frac{B}{r} e^{ik(ct+r)}, \tag{3·65}$$

where A and B are arbitrary constants and we must consider both converging and diverging waves. The condensation and excess pressure are given by

$$s = -\frac{\dot{\varphi}}{c^2} = -\frac{ik}{cr}\left[Ae^{ik(ct-r)} + Be^{ik(ct+r)}\right] \tag{3·66}$$

and

$$p = \rho_0 c^2 s = -\frac{ik\rho_0 c}{r}\left[Ae^{ik(ct-r)} + Be^{ik(ct+r)}\right]. \tag{3·67}$$

Now if the conical tube is *open* at both ends the excess pressure

will be approximately zero at $r = r_1$ and $r = r_2$ for all values of t. Recognizing that A and B are in general complex and writing $A = a_1 + ia_2$ and $B = b_1 + ib_2$, we finally have, using the real parts,

$$\begin{aligned}(a_1 + b_1)\cos kr_1 + (a_2 - b_2)\sin kr_1 &= 0,\\ (a_1 + b_1)\cos kr_2 + (a_2 - b_2)\sin kr_2 &= 0.\end{aligned} \tag{3·68}$$

These lead at once to

$$\sin k(r_2 - r_1) = 0$$

or

$$r_2 - r_1 = \frac{n\lambda}{2}, \tag{3·69}$$

wherein we note the interesting fact that if the conicality is slight, eq. (3·69) reduces to the condition already deduced for cylindrical pipes (Sec. 2·6). For a conical pipe with ends closed we must have $\dot{\xi} = 0$ at $r = r_1$ and $r = r_2$. Now

$$\dot{\xi} = \frac{\partial \varphi}{\partial r} = -\left(\frac{1}{r} + ik\right)\frac{A}{r}e^{ik(ct-r)} - \left(\frac{1}{r} - ik\right)\frac{B}{r}e^{ik(ct+r)}. \tag{3·70}$$

If neither r_1 nor r_2 is zero, we have the two equations

$$\begin{aligned}Ae^{-ikr_1}[1 + ikr_1] + Be^{+ikr_1}[1 - ikr_1] &= 0,\\ Ae^{-ikr_2}[1 + ikr_2] + Be^{+ikr_2}[1 - ikr_2] &= 0.\end{aligned} \tag{3·71}$$

Proceeding as usual, eqs. (3·71) become

$$\begin{aligned}(a_1+b_1)(\cos kr_1 + kr_1 \sin kr_1) + (a_2-b_2)(\sin kr_1 - kr_1 \cos kr_1) &= 0,\\ (a_1+b_1)(\cos kr_2 + kr_2 \sin kr_2) + (a_2-b_2)(\sin kr_2 - kr_2 \cos kr_2) &= 0,\end{aligned}$$

whence by elimination we have

$$kr_1 - \text{arc tan } kr_1 = kr_2 - \text{arc tan } kr_2, \tag{3·72}$$

a transcendental equation from which the relation between r_1, r_2 and λ must be evaluated graphically.

In the case of a cone closed at the vertex we shall have $r_1 = 0$, whence it would seem that the above reasoning could not apply. However, we know that at the origin of spherical waves $\dot{\xi}$ must be finite and therefore $r^2\dot{\xi}$ must approach zero as $r = 0$. For this reason the above equation (eq. (3·72)) still holds for $r_1 = 0$ and becomes (if we set $r_2 = r$ simply)

$$kr = \tan kr. \tag{3·73}$$

Graphical solution of eq. (3·73) will be found in Barton, Textbook of Sound, p. 261, Fig. 51.

Suppose next that we have a progressive spherical wave traveling from left to right through a pipe made of two cones, as in Fig. 3·4. It is desired to calculate the power transmission through the system. Let the distance from O_1 to the boundary circle be r_1 and that from O_2 be r_2. Then at the boundary we shall have an incident and reflected wave in the left section and a transmitted wave in the right section. The expressions for the displacement velocities at the boundary for these waves then are

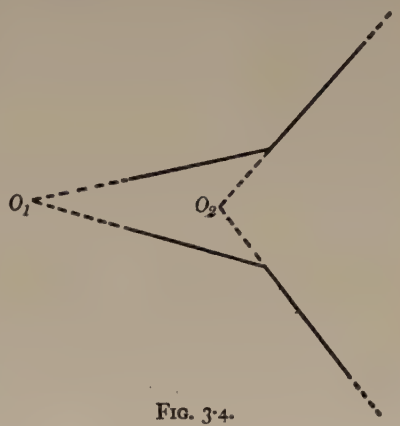

FIG. 3·4.

$$\begin{aligned}
\dot{\xi}_{i,\,r_1} &= -\left(\frac{1}{r_1} + ik\right)\frac{A_1'}{r_1}e^{ik(ct-r_1)} = -\left(\frac{1}{r_1} + ik\right)\frac{A_1}{r_1}e^{ikct}, \\
\dot{\xi}_{r,\,r_1} &= +\left(\frac{1}{r_1} - ik\right)\frac{B_1'}{r_1}e^{ik(ct+r_1)} = +\left(\frac{1}{r_1} - ik\right)\frac{B_1}{r_1}e^{ikct}, \qquad (3\cdot74) \\
\dot{\xi}_{t,\,r_2} &= -\left(\frac{1}{r_2} + ik\right)\frac{A_2'}{r_2}e^{ik(ct-r_2)} = -\left(\frac{1}{r_2} + ik\right)\frac{A_2}{r_2}e^{ikct},
\end{aligned}$$

where we have consolidated by writing $A_1 = A_1'e^{-ikr_1}$, etc., and where the plus sign in the expression for $\dot{\xi}_{r,\,r_1}$ is due to the fact that $\dot{\xi}_{r,\,r_1}$ is measured in the direction of decreasing r. For the excess pressures at the boundary we have similarly

$$\begin{aligned}
p_{i,\,r_1} &= \frac{-ik\rho_0 cA_1}{r_1}e^{ikct}, \\
p_{r,\,r_1} &= \frac{-ik\rho_0 cB_1}{r_1}e^{ikct}, \qquad (3\cdot75) \\
p_{t,\,r_2} &= \frac{-ik\rho_0 cA_2}{r_2}e^{ikct}.
\end{aligned}$$

The boundary conditions are as usual:

(1) continuity of pressure,

(2) continuity of volume displacement or current.

That is, we have

$$\frac{A_1}{r_1} + \frac{B_1}{r_1} = \frac{A_2}{r_2} \tag{3·76}$$

and

$$\left(\frac{1}{r_1} + ik\right)\frac{A_1}{r_1} + \left(\frac{1}{r_1} - ik\right)\frac{B_1}{r_1} = \left(\frac{1}{r_2} + ik\right)\frac{A_2}{r_2}. \tag{3·77}$$

By eliminating A_2 between the above two equations we arrive at the equation

$$B_1 = -A_1 \frac{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + 2ik}, \tag{3·78}$$

which, on introducing the transformation

$$\frac{1}{k}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = 2\gamma, \tag{3·79}$$

becomes

$$B_1 = A_1 \frac{\gamma(i - \gamma)}{1 + \gamma^2}, \tag{3·80}$$

whence we have at once from eq. (3·76)

$$\frac{A_2}{r_2} = \frac{A_1}{r_1}\left(\frac{1 + i\gamma}{1 + \gamma^2}\right). \tag{3·81}$$

Now the power transmission ratio is

$$P_r = \frac{S_{r_2}\overline{p_t\dot{\xi}_t}}{S_{r_1}\overline{p_i\dot{\xi}_i}}. \tag{3·82}$$

Here we have $S_{r_2} = S_{r_1}$. In forming the products of p and $\dot{\xi}$, we must, of course, take the real parts, writing

$$\begin{aligned} A_1 &= |A_1|e^{i\theta_1}, \\ A_2 &= |A_2|e^{i\theta_2}, \end{aligned} \tag{3·83}$$

where $|A_1|$ and $|A_2|$ are the moduli of A_1 and A_2 respectively. We then have, taking the time averages, the following result:

$$P_r = \left[\frac{|A_2|/r_2}{|A_1|/r_1}\right]^2. \tag{3·84}$$

By the application of eq. (3·81) above, this reduces at once to

$$P_r = \frac{1}{1+\gamma^2} = \frac{1}{1+\frac{1}{4}\left(\frac{\lambda}{2\pi}\right)^2\left(\frac{1}{r_2}-\frac{1}{r_1}\right)^2} \tag{3·85}$$

in terms of the wave length and the radii. It is worth noting that in this kind of transmission for a given system the yield increases greatly as the frequency increases,[1] and there is otherwise no selective effect such as we found in Section 3·6. When $r_2 = r_1$, there is, of course, no change in the conicality, and 100% yield results (always neglecting dissipation) for all frequencies.

It is obvious that the assumption is made that the wave is always spherical and that at the change in conicality spherical waves with two radii of curvature of wave fronts exist at the same point.

It is interesting also to compare eq. (3·85) with P_r for the case where a plane wave in a tube encounters an abrupt change in cross section (as in Sec. 3·5). In the latter case (see eq. (3·49)) the power transmission ratio depends only on the change in cross section and not at all on the wave length of the sound. In eq. (3·85) there is evident a dependence both on the dimensions of the tube and on wave length.

3·8. Conical and Exponential Connectors.—In Section 3·5 we have seen that an abrupt change in area of a tube through which sound is passing always leads to a decrease in the transmission. It is natural to suppose that if the transition from large cross section to small or vice versa is made to occur more gradually, better transmission will be obtained. This is in fact the case as the following analysis shows.

Let us first consider the case of a conical connector, shown in diagrammatic form in Fig. 3·5.

There are here shown two conduits of cross sectional areas S_1 and S_2 respectively joined by the frustrum of a cone of slant height $r_2 - r_1 = l'$ and altitude $= l$. If a plane sound wave traverses the system from left to right, we shall have, at the boundary I,

[1] This obviously can not be extended to the limit of very high frequencies, for our fundamental theory does not contemplate wave lengths small compared with the transverse dimensions of the tubes.

incident and reflected *plane* waves to the left, and incident and reflected *spherical* waves to the right. At the boundary *II* we shall have incident and reflected spherical waves to the left and a transmitted plane wave to the right. The situation is formally much like that of the case of a constriction or expansion in a con-

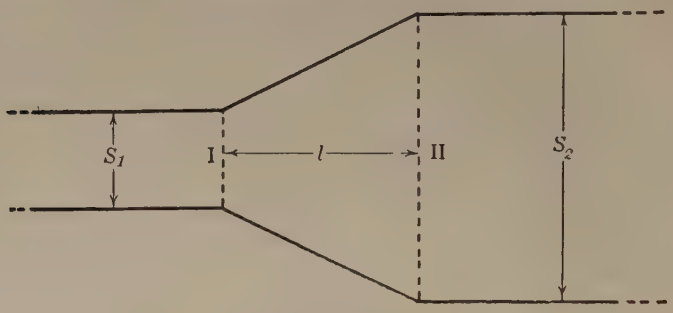

FIG. 3·5.

duit, already treated in Section 3·6; only here we have to consider the passage from plane to spherical waves and back again. It is to be observed, of course, that the assumed spherical or plane shape of the waves at these boundaries does not *strictly* exist, and that the theory is consequently only approximate.

For the incident and reflected wave *velocities* at I in the conduit let us take respectively $Ae^{i\omega t}$ and $Be^{i\omega t}$, while the corresponding quantities at I in the connector are (see eq. (3·74) and note that the factor $-ik$ is included in A_1 and B_1 here)

$$\left(1 - \frac{i}{kr_1}\right)\frac{A_1}{r_1}e^{i\omega t} \qquad \text{and} \qquad \left(1 + \frac{i}{kr_1}\right)\frac{B_1}{r_1}e^{i\omega t}.$$

The corresponding pressures at I are: in the conduit

$$\rho_0 c A e^{i\omega t} \qquad \text{and} \qquad \rho_0 c B e^{i\omega t},$$

while in the connector they are (see eq. (3·75))

$$\frac{\rho_0 c A_1}{r_1}e^{i\omega t} \qquad \text{and} \qquad \frac{\rho_0 c B_1}{r_1}e^{i\omega t}.$$

At the boundary *II* we have: in the connector

(1) Velocities:

$$\left(1 - \frac{i}{kr_2}\right)\frac{A_1}{r_2}e^{-ikl'}e^{i\omega t} \qquad \text{and} \qquad \left(1 + \frac{i}{kr_2}\right)\frac{B_1}{r_2}e^{ikl'}e^{i\omega t},$$

(2) Pressures:

$$\rho_0 c \frac{A_1}{r_2} e^{-ikl'} e^{i\omega t} \quad \text{and} \quad \rho_0 c \frac{B_1}{r_2} e^{-ikl'} e^{i\omega t}.$$

At *II* in the conduit we have:

(1) Transmitted velocity:

$$A_2 e^{i\omega t},$$

(2) Transmitted pressure:

$$\rho_0 c A_2 e^{i\omega t}.$$

We now have to write down the boundary conditions. At *I*, these become

$$A + B = \frac{1}{r_1}(A_1 + B_1), \tag{3·86}$$

$$A - B = \left(1 - \frac{i}{kr_1}\right)\frac{A_1}{r_1} - \left(1 + \frac{i}{kr_1}\right)\frac{B_1}{r_1}. \tag{3·87}$$

At *II*, they are

$$\frac{1}{r_2}(A_1 e^{-ikl'} + B_1 e^{ikl'}) = A_2, \tag{3·88}$$

$$\left(1 - \frac{i}{kr_2}\right)\frac{A_1}{r_2} e^{-ikl'} - \left(1 + \frac{i}{kr_2}\right)\frac{B_1}{r_2} e^{ikl'} = A_2. \tag{3·89}$$

The mathematical problem is to eliminate B, A_1 and B_1 from the above four equations and express A in terms of A_2. Adding the first two, we have

$$A = \frac{A_1}{2r_1}\left(2 - \frac{i}{kr_1}\right) - \frac{B_1}{2}\frac{i}{kr_1^2}. \tag{3·90}$$

Solving the last two for A_1 and B_1 in terms of A_2, we get

$$A_1 = \frac{A_2 r_2 e^{ikl'}}{2}\left(2 + \frac{i}{kr_2}\right), \tag{3·91}$$

$$B_2 = \frac{-A_2 r_2 e^{-ikl'}}{2} \cdot \frac{i}{kr_2}. \tag{3·92}$$

Substitution into eq. (3·90) then yields, after separation into real and imaginary parts,

$$A = \frac{r_2}{r_1} A_2 \left\{ \cos kl' + \frac{l'}{2kr_1r_2} \sin kl' + i\left[\left(1 + \frac{1}{2k^2 r_1 r_2}\right)\sin kl' - \frac{l'}{2kr_1r_2}\cos kl'\right]\right\}. \tag{3·93}$$

Noting that $S_2/S_1 = r_2^2/r_1^2$ and calling $\sqrt{S_2/S_1} = m$, and further introducing $\sigma = 2kl'$, whereby $r_1 = \sigma/2k(m-1)$ and $r_2 = (\sigma/2k) \times m/(m-1)$, we can express the coefficient of A_2 wholly in terms of m and σ. Thus

$$A = mA_2 \left\{ \cos\frac{\sigma}{2} + \frac{(m-1)^2}{m\sigma} \sin\frac{\sigma}{2} + i\left[\left(1 + \frac{2(m-1)^2}{m\sigma^2} \right) \sin\frac{\sigma}{2} - \frac{(m-1)^2}{m\sigma} \cos\frac{\sigma}{2} \right] \right\}. \quad (3\cdot94)$$

Now the power transmission ratio is

$$P_r = m^2 \frac{|A_2|^2}{|A|^2}. \quad (3\cdot95)$$

Hence we have

$$P_r = \frac{1}{1 + \left[\frac{(m-1)^2}{m\sigma}\right]^2 + \frac{4(m-1)^2}{m\sigma^2} \sin^2\frac{\sigma}{2} \cdot \left[1 + \frac{(m-1)^2}{m\sigma^2}\right] - \frac{4(m-1)^4}{m^2\sigma^3} \cos\frac{\sigma}{2} \sin\frac{\sigma}{2}}. \quad (3\cdot96)$$

This can be contracted into the somewhat simpler form

$$P_r = \frac{1}{\left[1 + \frac{(m-1)^2}{m} \frac{1-\cos\sigma}{\sigma^2}\right]^2 + \left[\frac{(m-1)^2}{m}\left(\frac{\sigma - \sin\sigma}{\sigma^2}\right)\right]^2}. \quad (3\cdot97)$$

by the use of trigonometric identities.

Brillié,[1] from whose articles the general outline of the above discussion is taken, has made computations using eq. (3·97) with results summarized in Fig. 3·6, showing the power transmission ratio for varying area ratios for sound of frequency 700 cycles. It is of especially great interest to note the greater yield in the case of the conical connection than in the corresponding abrupt change. It is important to remember, in this connection, that often the actual power transmission is not of such moment as the absence of a reflected wave.

Figure 3·6*a* shows in a somewhat more extensive fashion the power transmission ratio for conical connectors of various dimensions as a function of the frequency. The curves are a result of

[1] See Brillié, Le Génie Civil, 75, 218, 1919.

calculations made by one of the authors [1] by a method somewhat different from that used by Brillié (see below).

In Section 3·6 we saw (eq. (3·62)) that if a pipe has an intermediate channel and if the channel length is such that $\sin kl = \pm 1$,

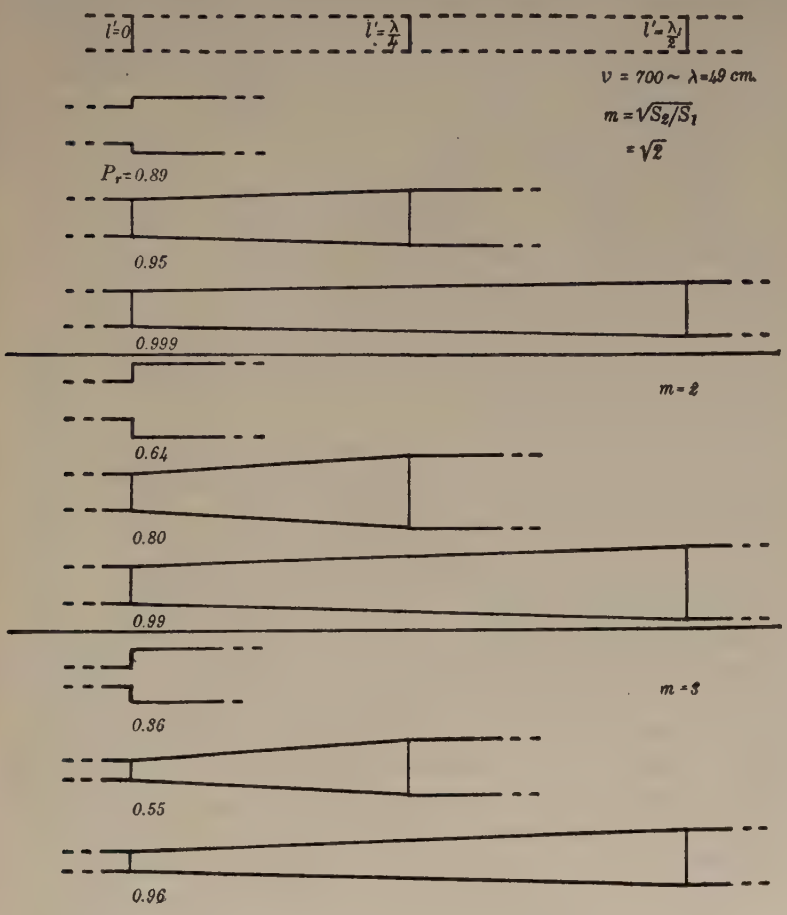

Fig. 3·6.

the transmission ratio is unity, if the cross section of the channel is a mean proportional between the areas of the pipe on the two sides of the channel. A number of channels each with cross section greater or smaller than the one preceding in accordance with the

[1] R. B. Lindsay, Phys. Rev., 34, 808, 1929.

proportion stated might be joined to form a connection between two tubes of different areas, and if the above condition is satisfied for each channel, $P_r = 1$ may be expected. If we could carry this to the limit of very small channel length the resulting connector will have a profile of exponential form, or $S = S_0 e^{mx}$. One might

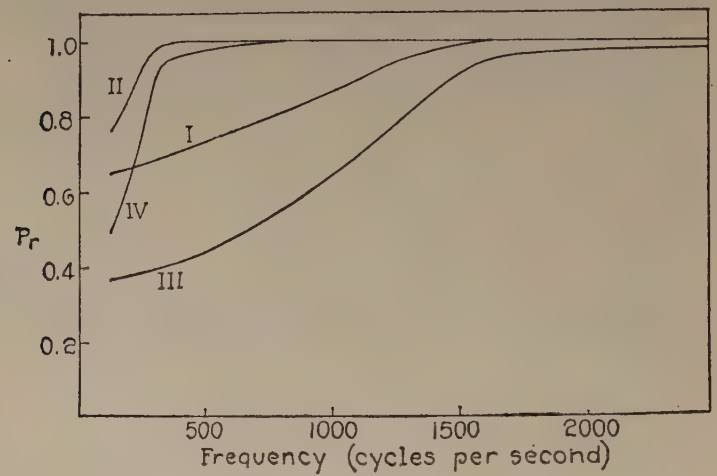

FIG. 3·6a. Power transmission through conical connector.
I, $m = 2$, $l = 10$ cm; II, $m = 2$, $l = 50$ cm.
III, $m = 3$, $l = 10$ cm; IV, $m = 3$, $l = 50$ cm.

expect then that the exponential connector will always give perfect transmission, but this is actually not true, and the trouble is that we are not logically justified in pushing the argument in the case of the discontinuous change in cross section to the continuous case.

As a matter of fact, if we treat the wave motion in the exponential connector as if the latter were an exponential horn, using the theory to be developed in Chapter VI, and then apply the usual boundary conditions, we arrive at the following expression for the transmission through an exponential connector:

$$P_r = \frac{1}{1 + \dfrac{m^2}{4(k^2 - m^2/4)} \sin^2 \gamma l}, \qquad (3{\cdot}98)$$

where m is the coefficient of x in the exponential and $\gamma^2 = k^2 - m^2/4$. It is seen that for very short connectors the value approaches unity. Naturally the validity of this formula rests on the validity of the horn theory used. A thorough discussion of

this, particularly as regards the exponential horn, will be found in Section 6·4.

Brillié [1] has further discussed the exponential connector as one through which the wave progresses without reflection. But this reasoning is based on the assumption of spherical waves, at best only approximate.

The results of the application of horn theory to acoustical connectors may be briefly summarized here.[2] Application to the case of the conical connector yields very simply essentially the results of the present section. In addition, the general case, where $S = S_0 x^a$ (a arbitrary), is worked out, and it is found that the transmission for kl large (l being here the *length* of the connector measured along the axis) approaches a value approximately independent of a. It is shown that the exponential connector is the limiting case of the above with increasing a and has a transmission ratio differing little from that of the former. Finally even in the case of a connector whose generating curve has a point of inflection, such as, for example, $S = S_0 e^{-ax^2}$, for large kl, the transmission shows little difference from that of the others. As kl decreases all these connectors yield values of P_r approaching that for abrupt change in cross section. These theoretical results are in agreement with the experimental data obtained by one of the authors,[3] according to which little if any difference could be detected between the transmission for actual models of the various types of connectors here discussed. Hence we may conclude that the exact shape of the connector is under usual conditions relatively unimportant.

3·9. Application of the Reciprocal Relation to Transmission in Conduits.—There is an important general principle about the transmission of sound through a conduit which we can establish by the use of the Helmholtz reciprocal theorem discussed in Section 1·11. Consider the tube of cross sectional area S_1, joined by a connector of *arbitrary* character with the tube of cross sectional area S_2. Suppose that there is a plane wave in S_1 of a displacement amplitude A_1 giving rise to a plane wave in S_2 of displacement amplitude A_2. Then the power transmission ratio for a wave going from left to right is

$$P_r = \frac{S_2 |A_2|^2}{S_1 |A_1|^2}. \qquad (3{\cdot}99)$$

[1] Brillié, Le Génie Civil, 75, 223, 1919.

[2] See R. B. Lindsay, loc. cit.

[3] G. W. Stewart, unpublished.

Suppose now we have plane waves from an *equal* source coming from S_2 in the direction from right to left. The amplitude this will produce in S_2 will be S_1A_1/S_2 since the maximum volume displacements due to equal sources are the same. Now according

FIG. 3·7.

to the reciprocal theorem the effect of the source in S_2 will be to produce the same velocity potential in S_1 as was produced in S_2 by an equal source in S_1. Since in this case the same is true of the displacement amplitude, the amplitude in S_1 is A_2. Therefore the power transmission ratio for transmission from right to left is

$$P_r' = \frac{S_1|A_2|^2}{S_2\left|\frac{S_1A_1}{S_2}\right|^2} = \frac{S_2|A_2|^2}{S_1|A_1|^2}. \qquad (3\cdot100)$$

Hence $P_r' = P_r$, and we conclude that no matter how the connection is made the yield is the same in either direction.

3·10. Transmission through Tubes in Parallel.—We may well close this chapter by a short discussion of sound transmission through parallel tubes. The simplest case of this kind is illustrated by the so-called Quincke Tube shown in the diagram (Fig. 3·8). S indicates the area of the tube and for simplicity we take $S_1 = S_2 + S_3$. A plane wave incident at the junction C divides in such a way that the following boundary conditions hold at C:

(1) Volume displacement in S_1 = Vol. displacement in S_2 + Vol. displacement in S_3,

(2) Pressure in S_1 = Pressure in S_2 = Pressure in S_3.

At C let us have the following for the incident and reflected wave displacements:

(*a*) in S_1 $\quad A_1e^{i\omega t}, \quad B_1e^{i\omega t},$

(*b*) in S_2 $\quad A_2e^{i\omega t}, \quad B_2e^{i\omega t},$

(*c*) in S_3 $\quad A_3e^{i\omega t}, \quad B_3e^{i\omega t}.$

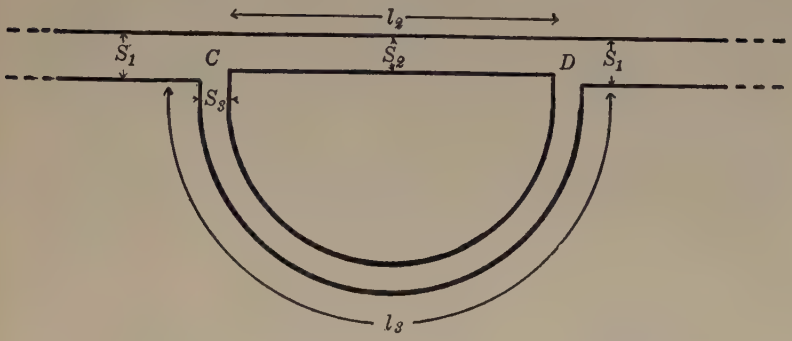

FIG. 3·8.

Then (referring back to Sec. 3·6) we have for the boundary conditions at C:

$$\begin{aligned} A_1 + B_1 &= A_2 + B_2 = A_3 + B_3, \\ S_1(A_1 - B_1) &= S_2(A_2 - B_2) + S_3(A_3 - B_3). \end{aligned} \tag{3.101}$$

If we denote the transmitted wave displacement at D by $A_4e^{i\omega t}$, the corresponding boundary conditions at D become

$$\begin{aligned} A_2e^{-ikl_2} + B_2e^{ikl_2} &= A_3e^{-ikl_3} + B_3e^{ikl_3} = A_4, \\ S_2(A_2e^{-ikl_2} - B_2e^{ikl_2}) + S_3(A_3e^{-ikl_3} - B_3e^{ikl_3}) &= S_1A_4. \end{aligned} \tag{3·102}$$

There are thus six boundary condition equations and the mathematical problem is to eliminate the five quantities B_1, A_2, B_2, A_3, B_3, and eventually express A_4 in terms of A_1. The power transmission will then be given at once by

$$P_r = \frac{|A_4|^2}{|A_1|^2}.$$

By carrying through the somewhat tedious elimination, we finally arrive at

$$P_r = \frac{16 \sin^2 \frac{k}{2}(l_3+l_2) \cos^2 \frac{k}{2}(l_3-l_2)}{[1-2\cos k(l_2+l_3)+\cos k(l_3-l_2)]^2+4\sin^2 k(l_3+l_2)}. \tag{3·103}$$

From the above equation it will be noted that $P_r = 0$ for the following cases:

$$(1)\ k(l_3 - l_2) = (2n+1)\pi \quad \text{or} \quad l_3 - l_2 = (2n+1)\frac{\lambda}{2} \qquad (3\cdot104)$$
$$(2)\ k(l_3 + l_2) = 2n\pi \quad \text{or} \quad l_3 + l_2 = n\lambda,$$

provided $k(l_3 - l_2) \neq 2n_1\pi$ at the same time, where n and n_1 are independent integers (including, of course, zero). These cases include the well known special case originally discussed by Quincke [1] in which the difference in length between the two tubes is one half wave length.

In the accompanying figure (Fig. 3·9) are shown the results of an actual experiment [2] on such a Quincke tube. It is interesting

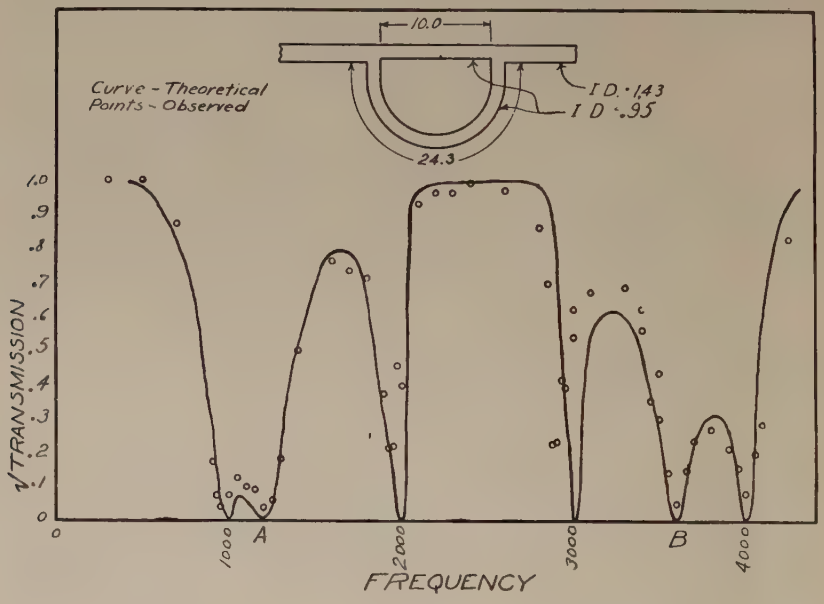

Fig. 3·9.

to note that the selectivity of such a tube is relatively sharp. This already suggests the possibility of sound filtration, which will be later discussed in Chapter VII. Moreover, the parallel tube is effectively a *branch* line, and this opens up the general problem of

[1] See Rayleigh, Theory of Sound, Vol. II, p. 63.

[2] G. W. Stewart, Phys. Rev. 31, 4, 696, 1928. It is interesting to note that the incorrect theory was in the literature for almost a hundred years.

transmission through a conduit with impedance in a branch, a topic exhaustively discussed in Chapter V.

Questions and Problems

1. Given a plane wave in a viscous fluid medium (take water) of infinite extent. If its frequency is 1000 cycles, how far will it travel before its amplitude is diminished in the ratio $1/e$? Solve the same problem if the frequency is 20,000 cycles. Work the same problem if the medium is air.

2. How would the results of Problem 1 come out if the same fluid were confined to a rigid tube 2 cm in diameter?

3. Derive the formula (eq. (3·21))

$$P = \frac{1}{2} p_{\max} \dot{\xi}_{\max} \cos\theta$$

for the average transmission per unit area.

4. Why cannot a discussion similar to that in Section 3·5 be used to determine the reflection at the *open* end of a conduit, i.e., where $S_1 = \infty$ in Fig. 3·1 and the wave is assumed to travel from right to left?

5. What conclusion may be derived from Section 3·6 as to the effectiveness of pinching a tube in order to reduce the intensity of transmission? Will the discussion of this section apply strictly to the case of a flat disc with a very minute hole inserted across an acoustic conduit?

6. Deduce the resonance conditions for a conical tube closed at the large end and open at the small end and compare with eqs. (3·69) and (3·73).

7. Carry through the calculation of the power transmission ratio for the Quincke-Herschel tube (Sec. 3·10).

8. Discuss the application of eq. (3·62) (Sec. 3·6) to an exponential connector and show where the analogy breaks down.

9. A piece of cheese cloth will stop the wind more effectively than it will a sound wave. Assume that the speed of the wind is 10 meters per second and that the sound wave of 500 cycles has a maximum excess pressure of 1 dyne/cm^2. Show by computation why the viscosity in the cheese cloth apertures is more highly effective in one case than in the other.

CHAPTER IV

Transmission.—Changes in Media

4·1. Change of Specific Acoustic Impedance at a Junction.—This section is devoted to the discussion of the transmission of sound energy across the junction of two media. Let OO' (Fig. 4·1) denote the boundary between the medium I of specific acoustic resistance (see Sec. 3·1) $R_1 = \rho_{01}c_1$ and medium II of specific resistance $R_2 = \rho_{02}c_2$. It will later be shown (Sec. 4·5) that for a plane longitudinal wave in a *solid* we can also define analogously the specific acoustic resistance. In this case also it proves to be

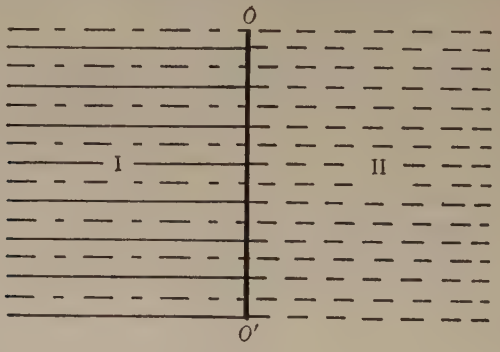

Fig. 4·1.

the product of density and wave velocity. Assume that there is a plane wave traveling from left to right from medium I to medium II. Let the wave displacements of the incident wave and reflected wave in I at the boundary be respectively

$$A_1 e^{i\omega t} \qquad \text{and} \qquad B_1 e^{i\omega t}$$

and that of the transmitted wave in II be

$$A_2 e^{i\omega t}.$$

The boundary conditions are, as usual,

(1) continuity of pressure,

(2) continuity of volume displacement or of volume current.

Consulting eqs. (3·41), (3·42), (3·43) and (3·44), Section 3·5, we have for these two conditions in the present case, respectively,

$$R_1(A_1 + B_1) = R_2 A_2 \tag{4·1}$$

$$A_1 - B_1 = A_2. \tag{4·2}$$

Solving, we find

$$B_1 = A_1 \frac{R_2 - R_1}{R_2 + R_1} = A_1 \frac{r_{12} - 1}{r_{12} + 1}, \tag{4·3}$$

$$A_2 = A_1 \frac{2R_1}{R_2 + R_1} = A_1 \frac{2}{r_{12} + 1}, \tag{4·4}$$

wherein we have introduced $r_{12} = R_2/R_1$ to denote the *relative* specific acoustic resistance of medium II with respect to medium I. (Note the analogy with the *reciprocal* of the relative index of refraction of two media.)

The energy flow in the incident wave is proportional to (see eqs. (1·47) and (3·60′) with accompanying discussion)

$$R_1 \omega^2 |A_1|^2,$$

while that in the transmitted wave is

$$R_2 \omega^2 |A_2|^2.$$

Therefore the power transmission ratio is

$$P_r = \frac{R_2}{R_1} \frac{|A_2|^2}{|A_1|^2} = \frac{4r_{12}}{(r_{12} + 1)^2}. \tag{4·5}$$

Referring to eq. (4·3) it is seen that the reflected wave is *in phase* with the incident wave (both as regards displacement and pressure) if $r_{12} > 1$, while the two waves are opposite in phase if $r_{12} < 1$. Incidentally these facts have already been deduced previously in Section 1·18, where the problem of reflection at a boundary was treated without regard to transmission. From eq. (4·4) we now see that the transmitted wave is *always* in phase with the incident wave. It will also be noted that if $r_{12} >> 1$ or $r_{12} << 1$, the amplitude of the reflected wave is approximately equal to that of the incident wave. In both cases practically all the incident energy is reflected, that is, the transmitted energy is nearly zero. In the first case the transmitted amplitude is very small, though in the second case the transmitted amplitude is approximately *twice* that of the incident amplitude. The latter case corresponds to passage from a very dense to a very rare medium.

The reader will recall (see Sec. 3·5) that the formula (4·5) is identical in form with eq. (3·49) for the power transmission through a tube with an abrupt change in cross section. The two would in fact be identical if we set formally $m = S_2/S_1 = r_{12} = R_2/R_1$. This formal analogy, which of course results from the similar boundary condition equations in the two cases, will facilitate the analytical treatment of the transmission problems of this chapter.

An extensive table of values of specific acoustic resistance will be found in Appendix I.

4·2. Amount of Reflection at a Boundary for Oblique Incidence. —In the preceding work the passage from one medium to another has been assumed of such a character that the incident wave front is parallel to the boundary. It will be of interest to examine briefly the more general case in which the incidence is not normal but where the glancing angle, θ_1, differs from zero. The construction of the refracted wave front has already been given in Section 1·8 (Fig. 1·7), and the law of refraction, namely, $\sin\theta_1/\sin\theta_2 = c_1/c_2$, where c_1 and c_2 are the velocities of sound in the two media respectively, still holds. If the incident medium is a fluid, we need to consider only the longitudinal wave in the second medium, because the liquid can exert no shearing stress.

The boundary conditions are of the same type as those we are already acquainted with, namely, continuity of pressure at the boundary and continuity of the component displacement or particle velocity normal to the boundary.

Let us consider a plane wave AB (Fig. 4·2) incident on the boundary, which we take here to be the YZ plane whose trace on the XY plane is of course the Y axis. The direction of travel is indicated by the arrows and the angle of incidence is θ_1. The reflected wave front is CD and the refracted wave front in the second medium is CE with angle of refraction θ_2. Strictly speaking these are the traces of the wave front, which is taken perpendicular to the XY plane. Let the original wave front passing through the fixed origin be denoted by

$$a_1x + b_1y = 0.$$

We may then denote the incident wave front AB by

$$a_1x + b_1y = K,$$

where the perpendicular distance between the two or the distance

the wave has traveled from the initial position is thus

$$d = \frac{K}{\sqrt{a_1^2 + b_1^2}} = \frac{a_1 x + b_1 y}{\sqrt{a_1^2 + b_1^2}}.$$

The expression for the velocity potential of the incident wave must be in the form

$$\varphi_i = A_1 e^{i[\omega t - k_1 d]},$$

where $k_1 = \omega/c_1 = 2\pi/\lambda_1$. Hence on substitution this becomes

$$\varphi_i = A_1 e^{i\{\omega t - k_1 (a_1 x + b_1 y)/\sqrt{a_1^2 + b_1^2}\}}.$$

Now since only the ratio $b_1/a_1 = \tan\theta_1$ is necessary to specify the direction of the incident wave front, the magnitude $\sqrt{a_1^2 + b_1^2}$

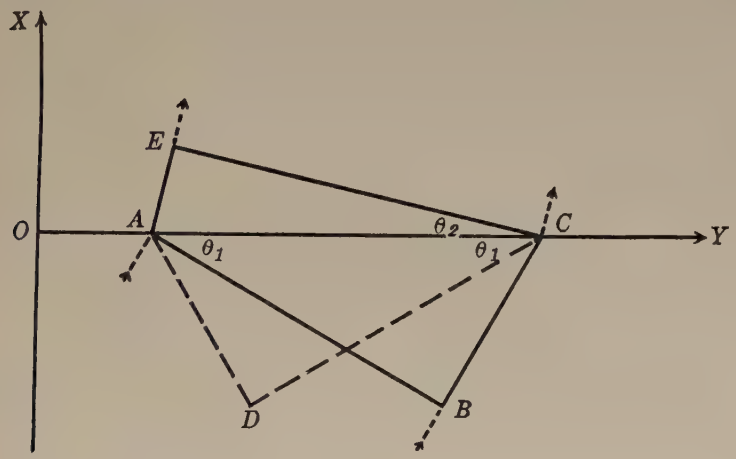

FIG. 4·2.

can be assigned any value we choose. It will be most simple to set $k_1 = \sqrt{a_1^2 + b_1^2}$, whence the above expression takes the form

$$\varphi_i = A_1 e^{i[\omega t - (a_1 x + b_1 y)]}.$$

Incidentally, substitution into the wave equation $\nabla^2 \varphi_i = \ddot{\varphi}_i / c^2$ shows that this is indeed a solution if $\sqrt{a_1^2 + b_1^2}$ be given the above value. We then have

$$a_1 = k_1 \cos\theta_1, \qquad b_1 = k_1 \sin\theta_1.$$

Now if we construct the reflected wave front by Huyghens' prin-

ciple, it is found that we must represent the reflected velocity potential by

$$\varphi_r = B_1 e^{i[\omega t + a_1 x - b_1 y]}.$$

This can be proved by actually determining geometrically the equation of the wave front, but it is simpler to note that from the diagram CD represents a wave front with components of motion in the positive y direction but negative x direction. Finally, the velocity potential in the refracted wave is represented by

$$\varphi_t = A_2 e^{i[\omega t - (a_2 x + b_1 y)]},$$

wherein in analogy with the incident case we have

$$a_2 = \sqrt{a_2^2 + b_1^2} \cos \theta_2 = k_2 \cos \theta_2$$

and also

$$b_1 = \sqrt{a_2^2 + b_1^2} \sin \theta_2 = k_2 \sin \theta_2.$$

In the expression for φ_t, the coefficient of y has been taken as b_1, as in φ_i and φ_r. The reason for this is that there must be no lateral slipping of the wave fronts at the boundary. Or we can also note that if we had assumed

$$\varphi_t = A_2 e^{i[\omega t - (a_2 x + b_2 y)]},$$

we snould have

$$a_2 = k_2 \cos \theta_2,$$
$$b_2 = k_2 \sin \theta_2.$$

But we have previously deduced

$$b_1 = k_1 \sin \theta_1.$$

Now from the law of refraction

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{c_1}{c_2} = \frac{k_2}{k_1}.$$

Hence $b_2 = b_1$. We now have to write down the boundary conditions. Since $p = -\rho_0 \dot{\varphi}$ (see eq. (1·13), Sec. 1·12), the pressure condition is

$$\rho_{01}\dot{\varphi}_i + \rho_{01}\dot{\varphi}_r = \rho_{02}\dot{\varphi}_t \qquad \text{at } x = 0. \tag{4·6}$$

Since $\dot{\xi} = u = \partial\varphi/\partial x$, the normal displacement velocity condition becomes

$$\frac{\partial \varphi_i}{\partial x} + \frac{\partial \varphi_r}{\partial x} = \frac{\partial \varphi_t}{\partial x} \qquad \text{at } x = 0. \tag{4·7}$$

These, on substitution, reduce to

$$\begin{aligned} \rho_{01}(A_1 + B_1) &= \rho_{02}A_2, \\ a_1(A_1 - B_1) &= a_2A_2, \end{aligned} \tag{4·8}$$

respectively. Solving these equations we have

$$A_1 = \frac{A_2(\rho_{02}a_1 + \rho_{01}a_2)}{2\rho_{01}a_1}, \tag{4·9}$$

$$B_1 = \frac{A_2(\rho_{02}a_1 - \rho_{01}a_2)}{2\rho_{01}a_1}, \tag{4·10}$$

whence

$$\frac{B_1}{A_1} = \frac{\rho_{02}a_1 - \rho_{01}a_2}{\rho_{02}a_1 + \rho_{01}a_2}. \tag{4·11}$$

But $a_1 = k_1 \cos\theta_1$ and $a_2 = k_2 \cos\theta_2$ as previously explained, and $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$. Recalling also the law of refraction we have

$$\frac{B_1}{A_1} = \frac{\rho_{02}/\rho_{01} - \cot\theta_2/\cot\theta_1}{\rho_{02}/\rho_{01} + \cot\theta_2/\cot\theta_1}. \tag{4·12}$$

Now the incident and reflected displacement amplitudes may be shown to be

$$\frac{A_1\sqrt{a_1^2 + b_1^2}}{kc} \quad \text{and} \quad \frac{B_1\sqrt{a_1^2 + b_1^2}}{kc} \quad \text{respectively.}$$

For if we denote the component displacements along the x and y directions by ξ_i and η_i, we find, since $\dot{\xi}_i = \partial\varphi_i/\partial x$ and $\dot{\eta}_i = \partial\varphi_i/\partial y$, that

$$\xi_i = -\frac{A_1a_1}{\omega}e^{i[\omega t - (a_1x + b_1y)]} \quad \text{and} \quad \eta_i = -\frac{A_1b_1}{\omega}e^{i[\omega t - (a_1x + b_1y)]}.$$

Hence the resultant displacement amplitude is

$$\sqrt{\xi_i^2 + \eta_i^2} = \frac{A_1\sqrt{a_1^2 + b_1^2}}{kc}.$$

Similarly for the reflected wave. It therefore follows that the ratio B_1/A_1 in eq. (4·12) is also the ratio of the reflected and incident displacement amplitudes. Consequently the fractional intensity reflected is simply

$$\left(\frac{B_1}{A_1}\right)^2 = \frac{[\rho_{02}/\rho_{01} - \cot\theta_2/\cot\theta_1]^2}{[\rho_{02}/\rho_{01} + \cot\theta_2/\cot\theta_1]^2}. \tag{4·13}$$

It is important to note that the frequency does not enter the above equation, whence we see that it is equally true for all frequencies.

For normal incidence $\theta_1 = 0$, and we have simply

$$\frac{B_1}{A_1} = \frac{\rho_{02}/\rho_{01} - c_1/c_2}{\rho_{02}/\rho_{01} + c_1/c_2}, \tag{4·14}$$

reducing to the same expression derived in Section 1·18 (eq. (1·64)). If the acoustic resistances of the two media are equal, i.e., if $\rho_{01}c_1 = \rho_{02}c_2$, then there is *no* reflected wave at all for normal incidence. In the more general case of oblique incidence, for the angle θ_1 such that

$$\frac{\cot\theta_2}{\cot\theta_1} = \frac{\rho_{02}}{\rho_{01}}, \tag{4·15}$$

we also have *no* reflected wave. Combining eq. (4·15) with the law of refraction, we find that this particular angle is given by

$$\cot^2\theta_1 = \frac{c_1^2/c_2^2 - 1}{\rho_{02}^2/\rho_{01}^2 - c_1^2/c_2^2}. \tag{4·16}$$

The existence of such an angle obviously depends on the condition that we have

$$\rho_{02}/\rho_{01} > c_1/c_2 > 1 \qquad \text{or} \qquad \rho_{02}/\rho_{01} < c_1/c_2 < 1,$$

that is, that c_1/c_2 is intermediate in value between unity and ρ_{02}/ρ_{01}.

Further notes on this subject will be found in Rayleigh, Vol. II, § 270. See also Chapter XII, Sections 12·1 and 12·3, for a discussion of total reflection in atmospheric acoustics.

4·3. Transmission through Three Media.—We now consider the normal transmission of sound through the three media I, II

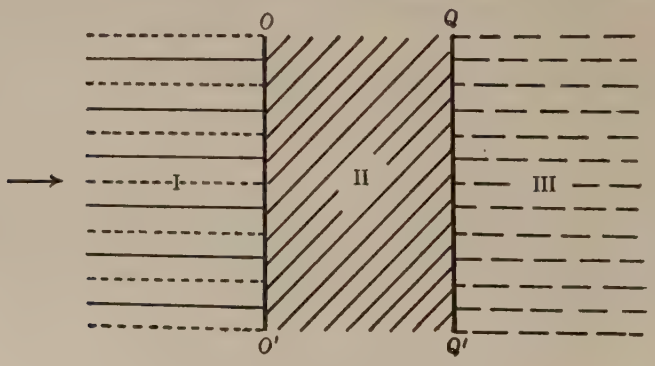

Fig. 4·3.

and III across the boundaries OO' and QQ'. With the three media will be associated the specific acoustic resistances R_1, R_2 and R_3 respectively. At the *first* boundary we shall have for the incident and reflected wave displacements in I:

$$A_1 e^{i\omega t} \quad \text{and} \quad B_1 e^{i\omega t},$$

while for the corresponding waves in II we have

$$A_2 e^{i\omega t} \quad \text{and} \quad B_2 e^{i\omega t}.$$

At the *second* boundary the corresponding waves are in medium II

$$A_2 e^{-ik_2 l} e^{i\omega t} \quad \text{and} \quad B_2 e^{ik_2 l} e^{i\omega t}$$

and the transmitted wave in medium III is

$$A_3 e^{i\omega t},$$

if l is the length of the second medium and $k_2 = \omega/c_2$, etc. Calling $r_{12} = R_2/R_1$ and $r_{23} = R_3/R_2$, we have at the *first* boundary the following:

(1) Pressure condition:

$$A_1 + B_1 = r_{12}(A_2 + B_2). \tag{4·17}$$

(2) Displacement condition:

$$A_1 - B_1 = A_2 - B_2. \tag{4·18}$$

At the *second* boundary we have:

(1) Pressure condition:

$$A_2 e^{-ik_2 l} + B_2 e^{ik_2 l} = r_{23} A_3. \tag{4·19}$$

(2) Displacement condition:

$$A_2 e^{-ik_2 l} - B_2 e^{ik_2 l} = A_3. \tag{4·20}$$

The mathematical problem is the elimination of B_1, A_2, B_2, to express A_3 in terms of A_1 and so find the transmission ratio, which is

$$P_r = \frac{R_3 |A_3|^2}{R_1 |A_1|^2} = r_{12} r_{23} \frac{|A_3|^2}{|A_1|^2}. \tag{4·21}$$

It is unnecessary to carry through the analysis, for it is formally identical with that of Section 3·6. Hence the result will be eq.

(3·61) with the substitution $m_1 = r_{12}$, $m_2 = r_{23}$, $k = k_2$. We have, therefore,

$$P_r = \frac{4r_{12}r_{23}}{(r_{12}r_{23} + 1)^2} \cdot \frac{1}{1 - \frac{(r_{23}^2 - 1)(r_{12}^2 - 1)}{(r_{12}r_{23} + 1)^2} \cdot \sin^2 k_2 l}. \quad (4\cdot22)$$

A study of eq. (4·22) reveals the following:[1]

(1) If $\sin k_2 l = \sin 2\pi(l/\lambda_2)$ is nearly zero (where λ_2 is the wave length in the second medium), the power transmission is

$$P_r = \frac{4r_{12}r_{23}}{(r_{12}r_{23} + 1)^2} = \frac{4r_{13}}{(r_{13} + 1)^2}, \quad (4\cdot23)$$

indicating that in this case the transmission is independent of the properties of the second medium. We note again the interesting fact that the introduction of a third medium between two media is formally equivalent to the introduction of an appropriate channel in a pipe. This means that all that was said in Section 3·6 can be formally applied here.

(2) If $\sin k_2 l \neq 0$, the yield depends on the intermediate medium only in the term $(r_{23}^2 - 1)(r_{12}^2 - 1)$ and we see that the presence of this medium will *increase* the transmission only if $(r_{12}^2 - 1) \times (r_{23}^2 - 1) > 0$. This means that $R_3 > R_2 > R_1$ or $R_3 < R_2 < R_1$, i.e., that the specific acoustic resistance of the intervening medium must lie between those of the other two media. Otherwise, the transmission is decreased.

(3) If $\sin k_2 l = 1$, the power transmission becomes (see eq. (3·62))

$$P_r = \frac{4r_{12}r_{23}}{(r_{12} + r_{23})^2}, \quad (4\cdot24)$$

and this is unity if $r_{12} = r_{23}$ or $R_1 R_3 = R_2^2$. The general statement also follows that for any three *given* media the yield is a maximum or a minimum for $\sin k_2 l = 1$ or $l = (2n + 1)\lambda_2/4$, where n is any integer (or zero). And indeed inspection of eq. (4·22) shows that we have a maximum under these conditions if $R_3 > R_2 > R_1$ and a minimum otherwise.

As an illustration, we may consider the transmission from water to air. Here we have, for water, $R_1 = 1.43 \times 10^5$ gm/cm² sec,

[1] Eq. (4·22) and the essential equations of Sections 4·4 and 4·6 have been deduced by H. Brillié, Le Génie Civil, **75**, pp. 194, 218, 1919.

while $R_3 = 42.6$ in the same units for air. Hence $r_{13} = 42.6/(1.43 \times 10^5) = 29.8 \times 10^{-5}$. Hence without any intermediate medium the transmission is only (see Sec. 4·1)

$$\frac{4r_{13}}{(r_{13}+1)^2} = 1.19 \times 10^{-3}.$$

By using an intervening medium of specific acoustic resistance $R_2 = \sqrt{R_1 R_3}$ this transmission can be brought up to unity for wave lengths $\lambda = 4l/(2n+1)$ (n any integer). Thus a glance at the table of values in Appendix I shows that if rubber were used as the intermediate medium this state of affairs would hold true approximately. Of course the assumption has to be made that the rubber vibrates like a true medium. If it does not, as it would not if it were short in length, a membrane or plate, for example, the argument above given is invalid. Whenever the medium vibrates as a whole, the above method fails of application. For the statement made above that when $\sin k_2 l = 0$ the transmission is independent of the properties of the intervening medium will be no longer true.

The problem of transmission of sound through two or three media is of obvious importance in subaqueous signalling and the results of this section will be used in the discussion in Chapter X.

4·4. The Stethoscope.—The results of the previous sections of this chapter and Chapter III may now be applied in the consideration of the stethoscope, shown in generalized form in Fig. 4·4. The

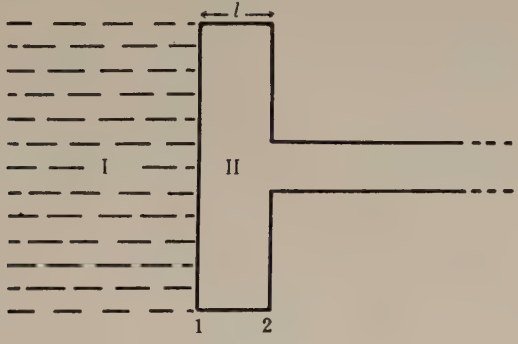

FIG. 4·4.

area of the broad base is S_1 and that of the small connecting tube S_2. The stethoscope medium is called II and it is exposed to a

medium I, the specific acoustic resistances being R_2 and R_1, respectively. As usual we denote the incident and reflected wave displacements in I at boundary 1 by $A_1e^{i\omega t}$ and $B_1e^{i\omega t}$ respectively, while the corresponding quantities in II at boundary 1 are $A_2e^{i\omega t}$ and $B_2e^{i\omega t}$. The transmitted wave at 2 is $A_3e^{i\omega t}$. The application of the usual boundary conditions then yields:

Boundary 1:

(1) Continuity of pressure:

$$A_1 + B_1 = r_{12}(A_2 + B_2). \tag{4·25}$$

(2) Continuity of displacement:

$$A_1 - B_1 = A_2 - B_2. \tag{4·26}$$

Boundary 2:

(1) Continuity of pressure:

$$A_2e^{-ik_2l} + B_2e^{ik_2l} = A_2. \tag{4·27}$$

(2) Continuity of displacement:

$$A_2e^{-ik_2l} - B_2e^{ik_2l} = \frac{S_2}{S_1}A_3, \tag{4·28}$$

wherein l is the thickness of the chamber at the base and $k_2 = 2\pi/\lambda_2$, where λ_2 is the wave length in the stethoscope medium.

By the usual analysis we obtain finally

$$\begin{aligned} P_r &= m_1r_{12}\frac{|A_3|^2}{|A_1|^2} \\ &= \frac{4m_1r_{12}}{(m_1 + r_{12})^2} \cdot \frac{1}{1 + \dfrac{(1 - m_1^2)(1 - r_{12}^2)}{(m_1 + r_{12})^2}\sin^2 k_2l}, \end{aligned} \tag{4·29}$$

where $m_1 = S_2/S_1$ as usual. There is a phase difference between the transmitted and incident wave. Thus if we write

$$A_1 = \frac{A_3}{2}e^{i\theta},$$

we have at once

$$\tan\theta = \frac{m_1r_{12} + 1}{r_{12} + m_1}\tan k_2l. \tag{4·30}$$

If two stethoscopic receivers are used together (as in the application of the binaural effect for direction finding, later to be explained), they must be matched for phase.

If $k_2 l$ is made very small, we have for the power transmission ratio

$$P_r = \frac{4m_1 r_{12}}{(m_1 + r_{12})^2} = 1 - \left(\frac{r_{12} - m_1}{r_{12} + m_1}\right)^2. \qquad (4{\cdot}31)$$

This becomes unity for $r_{12} = m_1$. Let us examine this numerically for the case of water as medium I, while the stethoscope medium is air. Then for unit yield we must have approximately

$$\frac{S_2}{S_1} = 3 \times 10^{-4},$$

which means a ratio of diameters of about one to 60. But the listening tube must be at least 3 or 4 mm in diameter and hence the area of the base of the stethoscope would have to be too large for practical purposes.

As a more practical illustration, let us consider a ratio of diameters of base and tube of 15 to one. Then $m_1 = 1/225 = 4.44 \times 10^{-3}$. Also $r_{12} = .3 \times 10^{-3}$. For a wave length λ of 44 cm, corresponding to a frequency of about 750 cycles at 0° C and normal pressure, and $l = 0.03$ cm, we find $P_r = .13$, or the transmission is about 13%. This thickness is rather small. If we take $l = 0.1$ cm we find for the same remaining data

$$P_r = .024,$$

or the transmission ratio of about 2.4%. This, of course, is rather small. But we may note that in the ordinary normal passage of sound from water to air without change in tube cross section the transmission ratio is only about 0.12% (see eq. (1·64), Sec. 1·18). Hence the use of the stethoscope makes an increase in sensitivity of some twenty times in the case cited. Here we again meet a principle of very great practical importance in subaqueous sound signalling.

It may be pointed out that the stethoscope as above described is actually an acoustic filter, and is indeed a special case of the general "tapered" transforming filter, recently discussed by W. P. Mason.[1]

4·5. Transmission of Sound in Solids.—It is often desirable to discuss the propagation of sound from liquid or gas to solids, as,

[1] W. P. Mason, The Bell System Technical Journal, VI, 258, 1927. See particularly part IV and compare with Chap. VII of the present work.

for example, the passage of sound from water to the hull of a ship. This demands some consideration of transmission in solid substances, in general a very complicated problem, since it requires a study of stress and strain in solid media. For the purpose of the discussion of the present section, however, we can simplify matters by considering merely the longitudinal motion along the axis of a rod, where the transverse motions may be safely neglected. Figure 4·5 shows a rod in the form of a rectangular parallelepiped with its length measured in the x direction. Consider a slab of

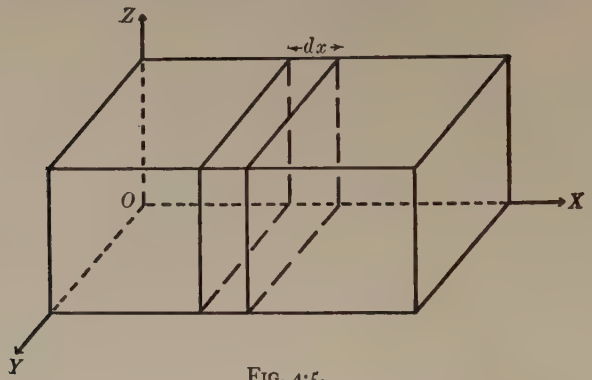

FIG. 4·5.

the rod of length dx at the distance x from O, and let the stress (force/area) in the x direction on the two faces be X_x and $X_x + (\partial X_x/\partial x)dx$, respectively. The net stress in the x direction is then $(\partial X_x/\partial x)dx$, and the total unbalanced force on the slab in the x direction is $(\partial X_x/\partial x)dxdydz$.

The equation of motion of the slab is then

$$\rho\ddot{\xi} = \frac{\partial X_x}{\partial x}.$$

Applying Hooke's law, we have

$$\frac{X_x}{\dfrac{\partial \xi}{\partial x}} = Y, \tag{4·32}$$

where Y is in this case Young's Modulus. Therefore the equation of motion becomes

$$\rho\ddot{\xi} = Y\frac{\partial^2 \xi}{\partial x^2}, \tag{4·33}$$

or the usual wave equation for propagation in the x direction with wave velocity $\sqrt{Y/\rho}$. Moreover we can define acoustic impedance for such a longitudinal wave in a solid in a fashion wholly analogous to that employed for waves in a fluid. This is a matter of some importance in view of our assumptions in Section 4·1. Thus, just as we defined the acoustic impedance in the case of a fluid (see Sec. 2·3) as pressure/volume current, the analogous quantity for longitudinal waves in a solid will be (negative stress)/area $\times$ displacement velocity $= (-X_x)/S\dot{\xi}$ in our notation above. Now from the equation of motion and (4·33) we have in the simple case just treated

$$\frac{\partial X_x}{\partial x} = Y\frac{\partial^2 \xi}{\partial x^2}$$

or

$$X_x = Y\frac{\partial \xi}{\partial x}. \qquad (4\cdot33a)$$

The solution to eq. (4·33) is of course for a single wave in the form

$$\xi = \xi_0 e^{i(\omega t - kx)},$$

whence

$$\frac{\partial \xi}{\partial x} = -ik\xi \qquad \text{and} \qquad \dot{\xi} = i\omega\xi.$$

Therefore (4·33a) yields at once

$$(-X_x)/S\dot{\xi} = \rho c/S, \qquad (4\cdot33b)$$

where $c = \sqrt{Y/\rho}$, the wave velocity. Comparison with eq. (3·5) shows that the acoustic impedance in this case is of exactly the same form as in that of a plane wave in a fluid, i.e., it is a resistance only. Moreover, we can then define

$$(-X_x)/\dot{\xi} = R = \rho c \qquad (4\cdot33c)$$

as the *specific acoustic resistance* of the longitudinal wave in the solid rod.

In Appendix II we shall develop the general theory of elastic waves in an isotropic solid. The chief results will be briefly set down here. Since such a solid has both shear elasticity and volume elasticity, a disturbance in it will in general be propagated by both longitudinal and transverse waves. If the modulus of volume elasticity is E and the shear modulus is n, the velocity of the longi-

tudinal wave in a large mass of the solid is found to be

$$C_L = \sqrt{\frac{E + 4/3 \cdot n}{\rho}}. \tag{4·34}$$

On the other hand the purely transverse waves set up by the torsion of a cylinder, for example, travel with the velocity

$$C_T = \sqrt{\frac{n}{\rho}}, \tag{4·35}$$

always considerably less than C_L.

The important thing to notice for our present purpose is that as long as we adhere to normal incidence we can discuss the transmission of sound from liquid or gas to solid in exactly the same way as we have treated the case of transmission between fluid media. In particular we now note the justification for the general statement of Section 1·18 that there is a phase change of zero or π when a compressional wave is reflected when normally incident at the interface of a solid and a fluid. For the fundamental boundary equations of that section hold here as well.

For a fluid the velocity reduces, as we have already noted in Section 1·12, to $c = \sqrt{E/\rho}$, and therefore the specific acoustic resistance in this case is $\sqrt{E\rho_0}$.

In connection with the above it is interesting to note that the seismic waves which occur in the earth are both longitudinal and transverse. At the earth's surface these two types have velocities which prove to be about 7.2 km/sec and 4 km/sec respectively, agreeing in a general way with the discussion in this section. Moreover there is a more or less uniform increase in the velocity of both types with the depth, so that at a depth of about one third the radius of the earth, the values become 12.7 and 6.8 km/sec. Thereafter they remain practically constant. For further details the reader should consult Horace Lamb, Science, Vol. LXII, No. 1602, 235, 1925.

4·6. Special Stethoscopes.—We are now in a position to investigate the action of a stethoscope in which a third medium is incorporated between the air and the water. For example this might be rubber. In the usual way we assume that the specific acoustic resistances of the three media are R_1, R_2 and R_3. Consulting Fig. 4·6, let the incident and reflected waves in I at boundary 1 be as usual

$$A_1 e^{i\omega t} \quad \text{and} \quad B_1 e^{i\omega t},$$

and in II at the same boundary let the corresponding waves be

$$A_2e^{i\omega t} \quad \text{and} \quad B_2e^{i\omega t}.$$

Also let the corresponding waves at boundary 2 in medium III be

$$A_3e^{i\omega t} \quad \text{and} \quad B_3e^{i\omega t}.$$

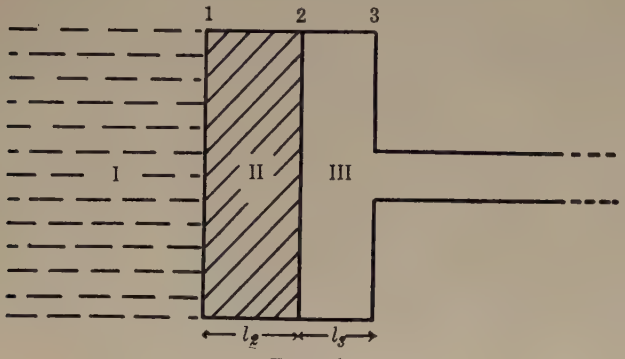

FIG. 4·6.

Then the boundary conditions, pressure and flow, may be written at once, as in the previous sections:

Boundary 1:

$$\begin{aligned} A_1 + B_1 &= (A_2 + B_2)r_{12}, \\ A_1 - B_1 &= A_2 - B_2. \end{aligned} \tag{4·36}$$

Boundary 2:

$$\begin{aligned} A_2e^{-ik_2l_2} + B_2e^{ik_2l_2} &= (A_3 + B_3)r_{23}, \\ A_2e^{-ik_2l_2} - B_2e^{ik_2l_2} &= A_3 - B_3. \end{aligned} \tag{4·37}$$

Boundary 3:

$$\begin{aligned} A_3e^{-ik_3l_3} + B_3e^{ik_3l_3} &= A_4, \\ A_3e^{-ik_3l_3} - B_3e^{ik_3l_3} &= m_1A_4, \end{aligned} \tag{4·38}$$

where $A_4e^{i\omega t}$ is the transmitted wave through the tube and $m_1 = S_2/S_1$, the ratio of tube cross section to that of the base of the stethoscope.

The mathematical problem, as usual, is to eliminate the five quantities B_1, A_2, B_2, A_3, B_3, and express A_1 in terms of A_4 to get the transmission ratio. Solving the last two equations for A_3 and B_3, we have

$$\begin{aligned} A_3 &= \frac{A_4(m_1 + 1)}{2}\, e^{ik_3l_3}, \\ B_3 &= \frac{A_4(1 - m_1)}{2}\, e^{-ik_3l_3}. \end{aligned} \tag{4·39}$$

Solving the second set, we have

$$A_2 = \frac{1}{2}[A_3(r_{23} + 1) + B_3(r_{23} - 1)]\cdot e^{ik_2l_2},$$
$$B_2 = \frac{1}{2}[A_3(r_{23} - 1) + B_3(r_{23} + 1)]\cdot e^{-ik_2l_2}. \tag{4·40}$$

Whence

$$A_2 = \frac{A_4}{4} e^{ik_2l_2}[(m_1 + 1)(r_{23} + 1)e^{ik_3l_3} + (1 - m_1)(r_{23} - 1)e^{-ik_3l_3}]$$

and

$$B_2 = \frac{A_4}{4} e^{-ik_2l_2}[(m_1 + 1)(r_{23} - 1)e^{ik_3l_3} + (1 - m_1)(r_{23} + 1)e^{-ik_3l_3}].$$

Finally solving the first two for A_1 and B_1,

$$A_1 = \frac{1}{2}[A_2(r_{12} + 1) + B_2(r_{12} - 1)],$$
$$B_1 = \frac{1}{2}[A_2(r_{12} - 1) + B_2(r_{12} + 1)], \tag{4·41}$$

or

$$\begin{aligned} A_1 = \frac{A_4}{8} e^{ik_2l_2}[&(m_1 + 1)(r_{23} + 1)e^{ik_3l_3} \\ &+ (1 - m_1)(r_{23} - 1)e^{-ik_3l_3}][r_{12} + 1] \\ + \frac{A_4}{8} e^{-ik_2l_2}[&(m_1 + 1)(r_{23} - 1)e^{ik_3l_3} \\ &+ (1 - m_1)(r_{23} + 1)e^{-ik_3l_3}][r_{12} - 1]. \end{aligned} \tag{4·42}$$

Let us now make the simplifying assumption that $k_2l_2 = (2n + 1)\cdot\pi/2$ where n is an integer, or $l_2 = (2n + 1)\cdot\lambda_2/4$, where λ_2 is the wave length in the second medium. Eq. (4·42) then reduces to

$$\begin{aligned} A_1 = \frac{A_4}{8} \{ & i(r_{12} + 1)[(m_1 + 1)(r_{23} + 1)e^{ik_3l_3} \\ &+ (1 - m_1)(r_{23} - 1)e^{-ik_3l_3}] \\ - & i(r_{12} - 1)[(m_1 + 1)(r_{23} - 1)e^{ik_3l_3} \\ &+ (1 - m_1)(r_{23} + 1)e^{-ik_3l_3}]\}. \end{aligned} \tag{4·43}$$

Then on further reduction

$$A_1 = \frac{A_4}{2} i[(m_1r_{12} + r_{23}) \cos k_3l_3 + i(r_{12} + m_1r_{23}) \sin k_3l_3]. \tag{4·44}$$

The power transmission ratio is

$$P_r = \frac{S_2 R_3 |A_4|^2}{S_1 R_1 |A_1|^2}$$

$$= \frac{4 m_1 r_{12} r_{23}}{(m_1 r_{12} + r_{23})^2} \cdot \frac{1}{1 + \frac{(1 - m_1^2)(r_{12}^2 - r_{23}^2)}{(m_1 r_{12} + r_{23})^2} \sin^2 k_3 l_3} \cdot \quad (4{\cdot}45)$$

Now if $\sin k_3 l_3$ is very small the power transmission ratio reduces to

$$P_r = \frac{4 m_1 r_{12} r_{23}}{(m_1 r_{12} + r_{23})^2}, \quad (4{\cdot}46)$$

and this equals unity for $m_1 r_{12} = r_{23}$, or

$$m_1 = \frac{S_2}{S_1} = \frac{R_3 R_1}{R_2^2}.$$

If for example the media I, II, III are water, rubber and air respectively, we have

$$\frac{S_2}{S_1} = \frac{(14) \times (.004) \times 10^8}{(.5)^2 \times 10^8} = .224,$$

whence the ratio of the diameters will be about 0.5 or as one is to two. It must not be forgotten, however, that the assumption $k_2 l_2 = \pi/2$ has introduced a selectivity, and it is only for certain wave lengths that we have the unit transmission. The same is true of the assumption that $\sin k_3 l_3$ is very small, though since l_3 is generally very small the selectivity introduced by this condition is not so pronounced.

It will be of interest to examine the value of P_r for the case where we do *not* assume $\sin k_2 l_2 = 1$, in order to see just how great is the selectivity introduced by this assumption. If we keep $e^{ik_2 l_2}$, etc., in the expression above, we finally obtain after considerable reduction

$$A_1 = \frac{A_4}{2}[C + iD], \quad (4{\cdot}47)$$

where

$$C = (r_{12} r_{23} + m_1) \cos k_2 l_2 \cos k_3 l_3 - (r_{12} + m_1 r_{23}) \sin k_2 l_2 \sin k_3 l_3,$$

and

$$D = (m_1 r_{12} + r_{23}) \sin k_2 l_2 \cos k_3 l_3 + (m_1 r_{12} r_{23} + 1) \sin k_3 l_3 \cos k_2 l_2.$$

The power transmission expression in its general form is rather complicated but for our purpose here it will be sufficient to develop it for the case where $k_3 l_3$ is very small, since that is the case of general interest. We then obtain

$$P_r = \frac{4 m_1 r_{12} r_{23}}{(m_1 r_{12} + r_{23})^2} \cdot \frac{1}{1 - \dfrac{(r_{12}^2 - 1)(m_1^2 - r_{23}^2)}{(m_1 r_{12} + r_{23})^2} \cos^2 k_2 l_2} \cdot \qquad (4{\cdot}48)$$

This expression reduces, of course, to eq. (4·46) for $\sin k_2 l_2 = 1$.

For the case of water-rubber-air under the optimum condition $m_1 r_{12} = t_{23}$ the formula reduces to

$$P_r = \frac{1}{1 + \dfrac{(r_{12}^2 - 1)^2}{4 r_{12}^2} \cos^2 k_2 l_2} \cdot \qquad (4{\cdot}49)$$

Now in this case we have

$$r_{12} = R_2/R_1 = .036,$$

whence

$$P_r = \frac{1}{1 + 192 \cos^2 k_2 l_2} \cdot$$

It is evident that, due to the large magnitude of the coefficient of $\cos^2 k_2 l_2$, the selectivity is very marked. Thus if we consider as a special case a thickness of rubber of 1.67 cm, while for a frequency of 750 cycles unit transmission occurs, for 600 cycles the transmission drops to about 5%.

It is clear that the special stethoscope described in the section will be of little advantage where a broad range of response is necessary. On the other hand, where fairly sharp and selective response is desirable it may prove extremely valuable. This point will be further emphasized in Chapter X in connection with submarine signalling.

Attention should again be called to the filtering characteristics of the stethoscope (see the last paragraph in Sec. 4·4).

Questions and Problems

1. In what respect does the discussion of a stethoscope fail if extended to unlimited areas?

2. Can the discussion of Section 4·3 include a medium II which is very thin?

3. Explain in physical terms the significance of the fact that, when a sound wave is incident normally from a dense medium to a rare medium, the transmitted amplitude is approximately *twice* the incident amplitude.

4. Find the condition for total reflection when a sound wave is incident on the interface of two media (Sec. 4·2).

5. From the discussion in Section 4·2 derive the expressions which are equivalent to Fresnel's equations for the relative intensity of light reflected from the interface of two media. Show that Brewster's law is satisfied (see Houstoun, Treatise on Light, 1915, p. 191).

6. Discuss the formula for the power transmission ratio for sound passing through three media (i.e., eq. (4·22)) by ascertaining the behavior of dP_r/dl and d^2P_r/dl^2 for various values of l. Repeat this for the corresponding stethoscope formula (eq. (4·29)).

7. By means of the relation between the elastic constants Y, E and n, express the velocity $c = \sqrt{Y/\rho}$ in terms of the velocities c_L and c_T (Sec. 4·5). Discuss from a physical point of view the difference between c and c_L.

CHAPTER V

Transmission through a Conduit with an Attached Branch or an Open End

5·1. General Theory of a Side Branch.—We now consider the general theory of the influence of a side branch on the transmission of sound through a conduit. In the figure (Fig. 5·1) there is rep-

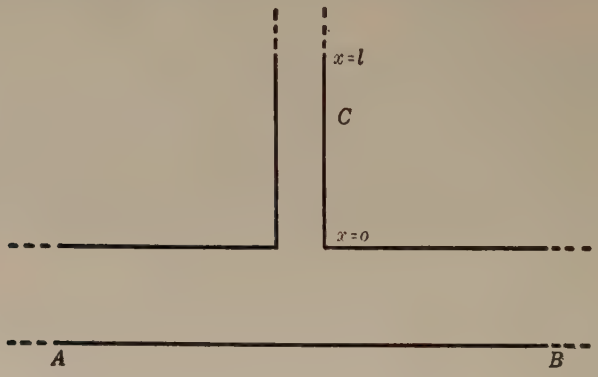

Fig. 5·1.

resented a conduit AB of uniform cross section S with a side branch C which may be open or closed and is arbitrary in nature. We have already discussed a special case of this general problem in the Quincke tube in Section 3·10. We wish to indicate here a general solution applicable to all cases.

Let the pressure in the plane wave incident from the left at the junction, where the branch is located, be $p_i e^{i\omega t}$. Let the reflected and transmitted pressures in the conduit at the same point be $p_r e^{i\omega t}$ and $p_t e^{i\omega t}$. Let the pressure in the branch at the junction, which is the same as $p_t e^{i\omega t}$, be represented for convenience as $p_b e^{i\omega t}$. As usual, the phase differences are taken care of by the possibility of p_r and p_t being complex. If we can find the ratio of p_t to p_i, we can get the power transmission simply as

$$P_r = \frac{|p_t|^2}{|p_i|^2}, \tag{5·1}$$

since the intensity in a plane wave may be expressed as $\overline{p^2}/\rho_0 c$ (Sec. 1·15) and the conduit suffers no change in cross section. The problem will be solved by setting up the boundary conditions expressing the continuity of pressure and volume current at the junction. For these we have

$$\begin{aligned} p_i + p_r &= p_b = p_t, \\ \dot{X}_i - \dot{X}_r &= \dot{X}_b + \dot{X}_t, \end{aligned} \tag{5·2}$$

if we denote volume current amplitudes by $\dot{X}$ with the appropriate subscripts. Now making use of the formula $p = \rho_0 c^2 s$ (Sec. 1·12) and the fact that for a plane wave $s = \dot{X}/Sc$, where S is the area of the wave front, and writing $Z_b = p_b/\dot{X}_b$, where Z_b is the impedance of the branch at the junction point, we finally have for the second of eqs. (5·2):

$$S/\rho_0 c \cdot (p_i - p_r) = p_b/Z_b + Sp_t/\rho_0 c, \tag{5·3}$$

which, since $p_b = p_t$, becomes

$$S/\rho_0 c \cdot (p_i - p_r) = p_t(1/Z_b + S/\rho_0 c).$$

Moreover, from the first of eqs. (5·2) we also have $p_i - p_r = 2p_i - p_t$, and hence on substitution there results

$$p_t/p_i = 1/(1 + \rho_0 c/2SZ_b). \tag{5·4}$$

The branch impedance will in general be complex; we therefore set $Z_b = Z_{b1} + iZ_{b2}$, and substitute into (5·4). Rationalization of the denominator and a simple reduction lead to the following form for the resulting power transmission:

$$P_r = \frac{Z_{b1}^2 + Z_{b2}^2}{[Z_{b2}^2 + (Z_{b1} + \rho_0 c/2S)^2]}. \tag{5·5}$$

This, then, is a general equation independent of the exact nature of the branch, and shows that an increase in $|Z_b|$ serves to increase P_r.

For a branch in which $Z_{b1} = 0$ (i.e., no acoustic resistance), the transmission reduces to

$$P_r = 1/(1 + \rho_0^2 c^2/4S^2 Z_{b2}^2), \tag{5·6}$$

while, if $Z_{b2} = 0$ (no acoustic reactance), we have

$$P_r = 1/(1 + \rho_0 c/SZ_{b1} + \rho_0^2 c^2/4S^2 Z_{b1}^2). \tag{5·7}$$

The general theory is illustrated by special cases in the following sections.[1]

5·2. The Helmholtz Resonator as a Branch.—As an application of the theory of Section 5·1 we proceed to consider the Helmholtz resonator as a side branch of an acoustic conduit, as in the accompanying figure (Fig. 5·2). There is no dissipation of acoustic energy

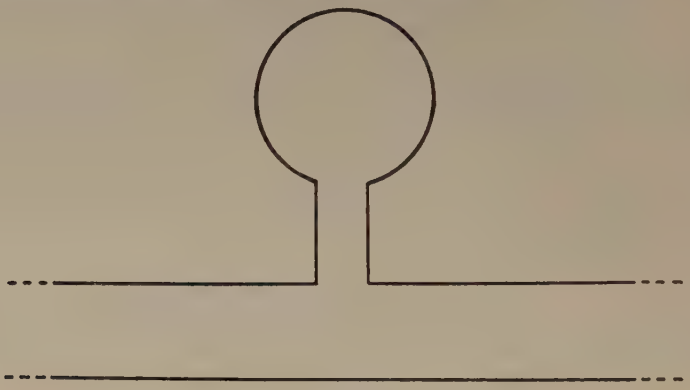

FIG. 5·2.

at the opening to the resonator, and therefore if we neglect viscosity, we have for the branch $Z_1 = 0$ and from Section 2·3 $Z_2 = M\omega - 1/\omega C$, where M is the inertance of the opening and C is the capacitance of the resonator chamber. From Section 2·3 we note that the inertance may be written as

$$M = \rho_0/c_0, \tag{5·8}$$

wherein c_0 is the "conductivity" of the opening. For a neck of length l and cross section S, this quantity should be (see Sec. 2·3)

$$c_0 = S/l. \tag{5·9}$$

But the actual mass of the moving fluid in the opening is never wholly included in the length l, and hence in general c_0 must be expressed as described in Section 2·4 (eq. (2·34) and accompanying discussion). This means that we shall have

$$c_0 = \frac{\pi a^2}{l + \beta a}, \tag{5·10}$$

[1] The theory of this section is a modification of that used by G. W. Stewart. See Phys. Rev., 26, 688, 1925. Equation 5 in this paper is algebraically equivalent to our more simple formula (5·5).

where β is a number which is usually between $\pi/4$ and $\pi/2$ depending on l. For an end opening into an infinite plane ($l = 0$), β approaches $\pi/2$ and $c_0 \doteq 2a$ for a circular orifice. Since $C = V/\rho_0 c^2$, where V is the volume of the chamber, we have for the power transmission ratio (Sec. 5·1, eq. (5·6))

$$P_r = \left[1 + \frac{\rho_0^2 c^2}{4S^2(\rho_0\omega/c_0 - \rho_0 c^2/V\omega)^2} \right]^{-1}. \qquad (5\cdot11)$$

In the above no attention has been paid to the influence of the viscosity in the orifice. In introducing this we shall make the assumption that the effect of the viscosity is equivalent to that in a channel of length l. Now if we have a viscous fluid moving in a tube of such a character that the layer adhering to the wall is small compared with the diameter, the equation of motion [1] may be written in the form

$$\rho_0 \ddot{X}\delta x + (1 + i)/a \cdot \sqrt{2\mu\rho_0\omega}\,\dot{X}\delta x = -S\partial p/\partial x \cdot \delta x, \qquad (5\cdot12)$$

wherein the first term on the left denotes the kinetic reaction of the fluid in a layer of thickness δx, the second term denotes the viscous drag on the fluid and the right hand side denotes the ordinary hydrostatic force due to the difference in pressure on the two sides of the layer. In this equation p is the mean pressure over the area inside the neck and X the mean volume displacement. The quantity μ is the coefficient of viscosity (see Sec. 3·3) and $S = \pi a^2$. Now if the length l is short, we may without great error assume that $\ddot{X}$ and $\partial p/\partial x$ are constant throughout the length of the channel, and for the total force per unit area on the gas in the neck we have

$$l\rho_0 \ddot{X}/\pi a^2 + l/\pi a^3 \cdot (1 + i)\sqrt{2\mu\rho_0\omega}\,\dot{X} = -l\partial p/\partial x. \qquad (5\cdot13)$$

We now have

$$-l\partial p/\partial x = p_b - \rho_0 c^2 X/V, \qquad (5\cdot14)$$

where p_b is the excess pressure at the orifice to the resonator (see the previous section) and $\rho_0 c^2 X/V$ is the pressure exerted by the resonator due to its acoustic capacitance $V/\rho_0 c^2$. The total impedance of the resonator is $p_b/\dot{X}$ and we may calculate it from eq. (5·13) if we recall that the volume displacement is of such a form that

$$\ddot{X} = i\omega\dot{X}. \qquad (5\cdot15)$$

[1] See Rayleigh, Theory of Sound, Vol. II, p. 318.

Substitution into eq. (5·13) then yields

$$[l/\pi a^3 \cdot \sqrt{2\mu\rho_0\omega} + i(l\rho_0\omega/\pi a^2 - \rho_0 c^2/\omega V + l/\pi a^3 \cdot \sqrt{2\mu\rho_0\omega})]\dot{X} = p_0. \quad (5\cdot16)$$

Whence we have for the impedance components

$$Z_1 = \text{acoustic resistance} = l/\pi a^3 \cdot \sqrt{2\mu\rho_0\omega}, \quad (5\cdot17)$$

$$Z_2 = \text{acoustic reactance} = \rho_0\omega/c_0 - \rho_0 c^2/V\omega + l/\pi a^3 \cdot \sqrt{2\mu\rho_0\omega}, \quad (5\cdot18)$$

if we substitute $c_0 = S/l$, the conductivity of the orifice of the resonator. From (5·13) and (5·18) we see that the acoustic resistance arises entirely from the viscosity and vanishes for $\mu = 0$, or for a sufficiently wide orifice in comparison with l, the length of the equivalent channel. So far as the acoustic reactance is concerned, we note (see eq. (2·20), Sec. 2·3) that the effect of the viscosity is to add a small term to the inertance of magnitude equal to the acoustic resistance or to subtract the same from the original capacitance of the resonator. To calculate the effect on the actual transmission it is necessary to substitute from (5·17) and (5·18) into eq. (5·5) of the preceding section. In the computation the one uncertain point lies in the value to be assigned to c_0, the conductivity of the orifice, and in any particular case the actual value of β to use in eq. (5·10) must be chosen with a certain unfortunate degree of arbitrariness.

The comparison of theoretical with experimental results has been made[1] in two cases with results indicated in Figs. 5·3 and 5·4. The first figure applies to the case of a conduit of cross sectional area 1.59 cm^2 with a branch resonator having an orifice 0.4 cm in diameter with a chamber volume of 35.7 cc. To calculate the conductivity c_0, the value $\beta = \pi/2$ is used, whence with $l = 0.015$ cm we get $c_0 = 0.382$ by the application of (5·10). The full curve in the figure is plotted using (5·11), since it is found on examination that the additional viscosity terms in the impedance will not alter the results materially for a fairly wide range of values of l in the vicinity of the actual length of the channel. Note that the orifice is here rather wide. The circles on the plot show the experimental values of the transmission, given at the left in terms of the usual fractional notation and at the right in the new decibel (*db*) notation [2]

[1] G. W. Stewart, Phys. Rev., **27**, 487, 1926.

[2] The transmission curves in the text to follow will in general express *Pr* in both notations, due to the growing importance of the latter scheme.

(see Sec. 1·15). The agreement is rather striking, showing as it does that the resonator seriously influences the transmission for more than an octave either side of the resonance frequency. This is the more surprising in view of the relatively great sharpness of the resonator response when used in the open. This is shown by the dotted curve in the figure, wherein the ordinates are the values of

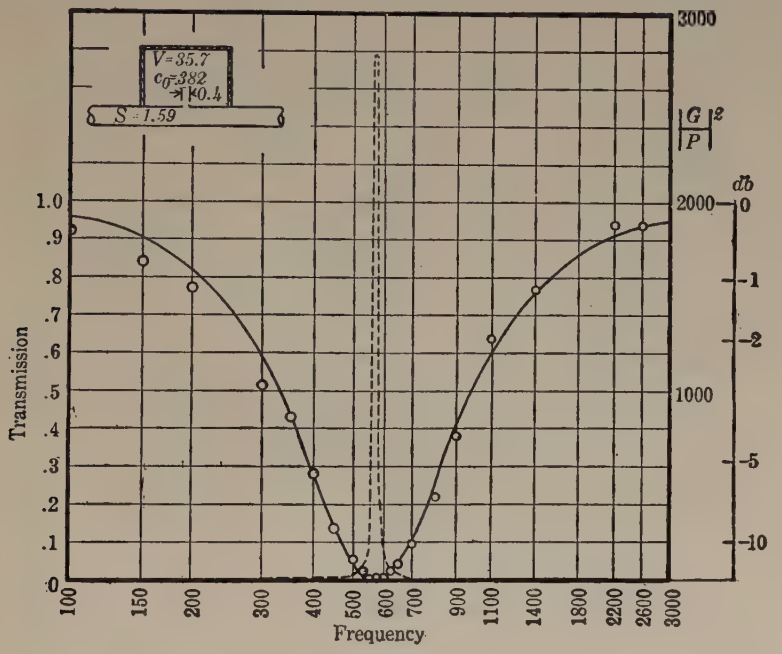

Fig. 5·3.

the amplification of the resonator as given by eq. (2·22) in Section 2·3. It is to be noted that the comparison of sharpness is made between two different measures of the effect of the resonator, both of which are in use.

To show the effect of viscosity a resonator with much smaller orifice was used with results indicated in Fig. 5·4. The chamber volume was reduced in order to bring the resonance frequency (without viscosity) to approximately the same value as in the previous case. Curve *a* is the curve obtained using eq. (5·11) (i.e., disregarding viscosity), while curve *b* is based on the more general expression (5·5) with the impedance components (5·17) and

(5·18) inserted. In the computation the value $\mu = 0.00018$ has been used and l has been taken as 0.09 cm or six times the actual length of the orifice. This seemed to be the value which best fitted the experimental results and a change of 10% results in a noticeable inferiority in the agreement. In any case the effect of the viscosity is well marked, particularly in the increased minimum transmission.

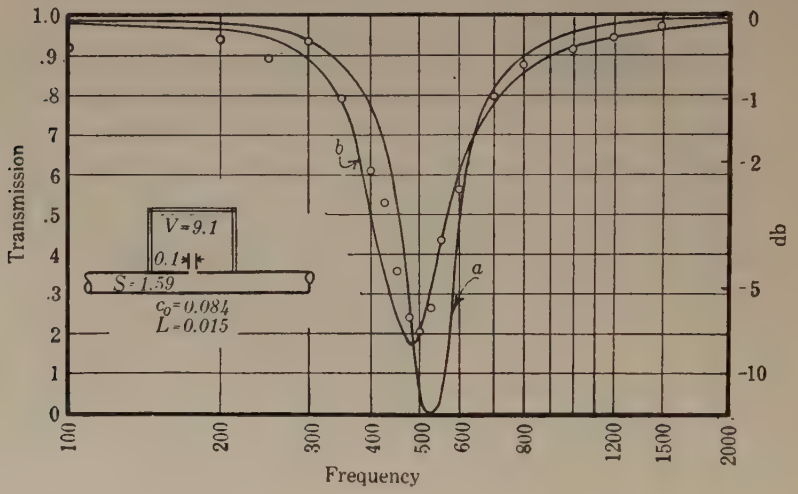

FIG. 5·4.

5·3. The Orifice as a Branch.—If the Helmholtz resonator is replaced by a simple orifice, we must modify the expression for the difference in pressure acting on the mass of air in the opening. That is, instead of $p_b - \rho_0 c^2 X/V$ (see eq. (5·14)), we have now the form $p_b - \rho_0 k\omega \dot{X}/2$, wherein the second term is the back pressure due to the radiation from the orifice (see Sec. 2·3, following eq. (2·13)). If now we substitute this quantity into the eq. (5·13), we have finally

$$[\rho_0\omega k/2\pi + \sqrt{2\mu\rho_0\omega}\cdot l/\pi a^3 + i(\rho_0\omega/c_0 + l\sqrt{2\mu\rho_0\omega}/\pi a^3)]\dot{X} = p_b, \qquad (5\cdot19)$$

whence for the impedance components we obtain

$$Z_1 = \rho_0\omega k/2\pi + l\sqrt{2\mu\rho_0\omega}/\pi a^3, \qquad (5\cdot20)$$

$$Z_2 = \rho_0\omega/c_0 + l\sqrt{2\mu\rho_0\omega}/\pi a^3. \qquad (5\cdot21)$$

This shows that the acoustic resistance is made up of a viscosity term and a radiation term. The acoustic reactance is now an inertance, but accompanied by a viscosity term also.

The results of experiments on the transmission with an orifice present are summarized in the accompanying Fig. 5·5. Four

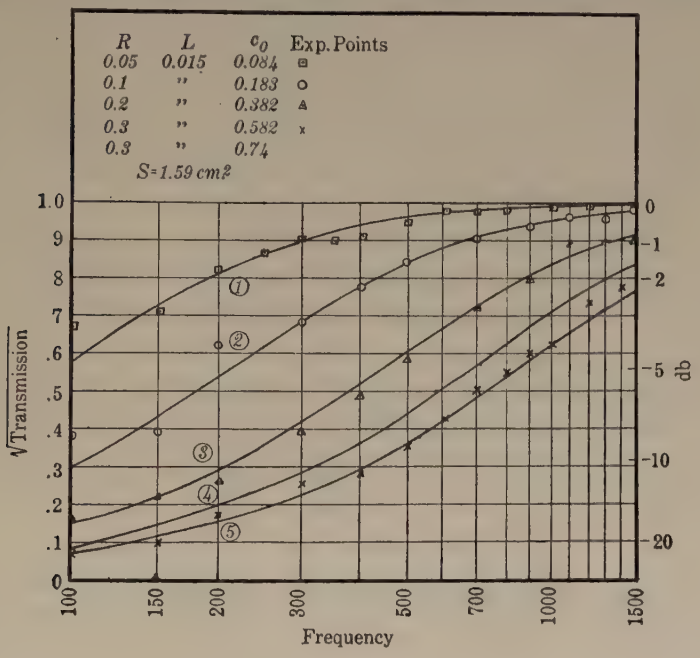

Fig. 5·5.

orifices were used [1] with radii running from 0.05 cm to 0.30 cm. The cross section of the conduit was 1.59 cm², and the channel length remained constant at 0.015 cm. The conductivities were calculated by the formula (5·10) already given above, and using $\beta = \pi/2$. It was found that applying the expressions for the impedance components to eq. (5·5), Section 5·1, the viscosity terms made very little difference in the result. From the curves it is seen that for reasonably small orifices the theory agrees well with experiment. But for larger orifices the agreement breaks down, as is evident from curve 4 in the figure. An arbitrary increase of the conductivity from 0.582 to 0.74 brings close agreement. This indicates

[1] G. W. Stewart, Phys. Rev., 27, 492, 1926.

that for large orifices the conductivity formula needs revision. Indeed the experiment may be looked upon as a means of calculating the conductivity in such cases.

To test the influence of the radiation from the orifice on the transmission the calculation was repeated, using $Z_1 = 0$ instead of the value given by eq. (5·24). The result (for the case of curve 5 in Fig. 5·5) is shown in the figure (Fig. 5·6) in the dotted curve.

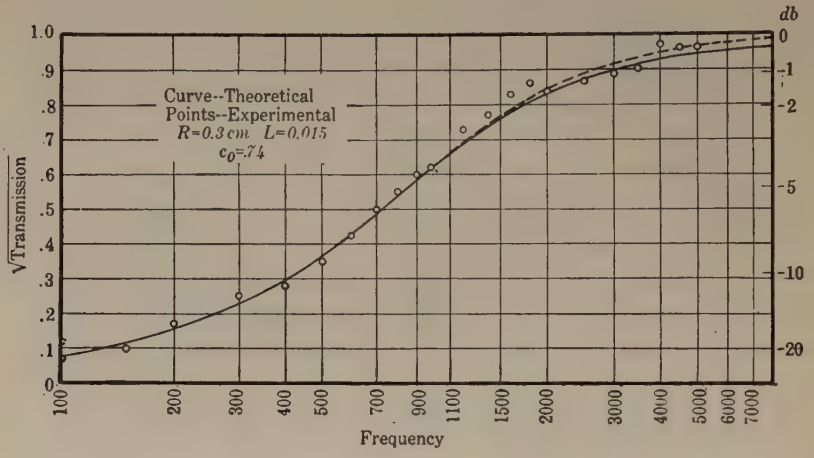

FIG. 5·6.

It is clear that the influence of the radiation is very small even at relatively high frequencies. This is, of course, what one would expect from the fact that

$$\frac{\rho_0 \omega k}{2\pi} < < \rho_0 \omega / c_0$$

for ordinary frequencies and particularly in the experiment here considered (see also Sec. 2·2).

We may summarize the results of the above discussion thus. First, orifices in a conduit diminish the transmission ratio, the diminution decreasing with increasing frequency of the sound. Second, neither viscosity nor radiation seriously affects the transmission in the cases cited. Third, the diminution in transmission is caused not so much by loss in energy in radiation as by the *inertance* of the orifice which produces a reflected wave. These last two results have an interesting bearing on the action of musical instruments having keys. Thus for an instrument with but one

control orifice (such as, for example, a clarinet with but one key lifted) much more of the energy *in a single wave* passing through the tube will emerge from the bell than from the orifice. But, of course, there will always be waves in both directions. Also we must not press the case too far, for a clarinet with several keys lifted constitutes in effect an acoustic filter (see Chap. VII), and in the latter case the sound emerging from the bell may be less than that from the holes; what is even more interesting, it may be of different quality. There is a great difference between the simple case of *transmission* and the more complicated case of *resonance* (i.e., that due to the reflected wave from the bell), which always plays a rôle in such instruments. The action of a clarinet, flute or similar instrument is very complicated and has not been quantitatively analyzed.

5·4. The Cylindrical Tube as a Branch.—It is the purpose of this section to derive the expression for the values of Z_1 and Z_2

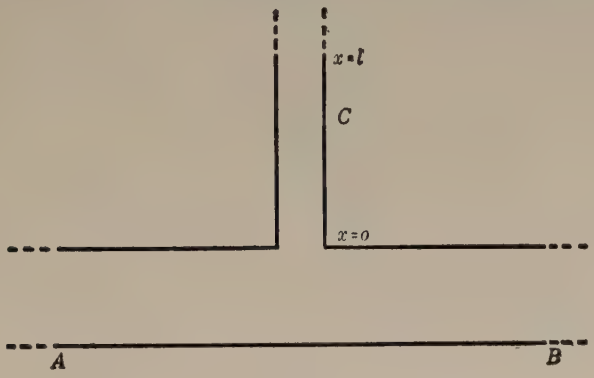

FIG. 5·7.

for any length of tubular side branch, open or closed. Substitution into eq. (5·5) of Section 5·1 will then enable us to compute the influence of such a tube on the transmission through the conduit. Let us consider the side tube C of constant cross section S. If we take the x axis along the axis of the tube, the equation of motion along the tube is, as usual,

$$\ddot{X} = c^2 \frac{\partial^2 X}{\partial x^2}, \qquad (5·22)$$

wherein $X = S\xi$. Since X is harmonic in the time with frequency

$\omega/2\pi$, and a steady state is assumed, the eq. (5·22) becomes

$$\frac{\partial^2 X}{\partial x^2} + k^2 X = 0, \tag{5·23}$$

where $k = \omega/c$ as usual. The general solution of (5·23) may be put into the form

$$X = (Ae^{ikx} + Be^{-ikx})e^{i\omega t} \tag{5·24}$$

corresponding to waves in both directions along the tube. A and B are the arbitrary constants, to be evaluated by the boundary conditions, and are in general complex quantities. From eqs. (1·14) and (1·26), $p = -\rho_0 c^2/S \cdot (\partial X/\partial x)$ and we have, omitting the variation with time,

$$p = -ik\rho_0 c^2/S \cdot Ae^{ikx} + ik\rho_0 c^2/S \cdot Be^{-ikx}. \tag{5·25}$$

If now (for simplicity) we let $x = 0$ at the point where the tube joins the conduit and $x = l$ at the other end (which may of course be open, closed or attached to some other conduit), we have the following four boundary equations:

$$\begin{aligned} X_0 &= A + B, \\ X_l &= Ae^{ikl} + Be^{-ikl}, \\ p_0 &= -ik\rho_0 c^2/S \cdot (A - B), \\ p_l &= -ik\rho_0 c^2/S \cdot (Ae^{ikl} - Be^{-ikl}). \end{aligned} \tag{5·26}$$

The complex impedance at the conduit end will be denoted by $Z_0 = p_0/\dot{X}_0$. That at the other end is $Z_l = p_l/\dot{X}_l$. The problem we are to solve is to express Z_0 in terms of Z_l and l, so that when these two are known, Z_0 may at once be obtained and the power transmission through the conduit evaluated. Since $\dot{X}_0 = i\omega X_0$, etc., we have from the first and third of eqs. (5·26)

$$Z_0 = -k\rho_0 c^2/\omega S \cdot \frac{(A/B - 1)}{(A/B + 1)}. \tag{5·27}$$

Moreover from the second and fourth equations there results

$$Z_l = -k\rho_0 c^2/\omega S \cdot \frac{(A/B \cdot e^{ikl} - e^{-ikl})}{(A/B \cdot e^{ikl} + e^{-ikl})}, \tag{5·28}$$

whence we solve for A/B and find

$$A/B = \frac{e^{-ikl}(-Z_l + k\rho_0 c^2/\omega S)}{e^{ikl}(Z_l + k\rho_0 c^2/\omega S)}. \tag{5·29}$$

Substitution into the eq. (5·27) for Z_0 now yields upon reduction

$$Z_0 = k\rho_0 c^2/S \cdot \left[\frac{i\omega Z_l \cos kl - k\rho_0 c^2/S \cdot \sin kl}{i\omega k\rho_0 c^2/S \cdot \cos kl - \omega^2 Z_l \sin kl} \right], \quad (5\cdot30)$$

which is the desired relation. When the tube is attached to the conduit as indicated, there must be added to the value of Z_0 above given the impedance of the orifice at the junction. In the case at hand this is a pure inertance and therefore has the value $i\rho_0\omega/c_0$, where c_0 is the conductivity of the orifice. The value of Z_l will obviously depend on the nature of the other end of the tube. If the tube is open and has an infinite flange, we shall have simply

$$Z_l = \rho_0\omega k/2\pi + i\rho_0\omega/c_{0l}, \quad (5\cdot31)$$

where $\rho_0\omega k/2\pi$ is the radiation resistance due to radiation from the opening, and c_{0l} is the conductivity of the opening (so denoted to distinguish it from c_0, the conductivity of the attached end of the tube). In this it will be noted that the frictional resistance due to viscosity is neglected. The justification for this procedure is obtained from the considerations of the preceding section.

If now we substitute into the expression for Z_0 above and write

$$Z_0 = Z_{01} + iZ_{02},$$

we have

$$Z_{01} = \rho_0\omega k/2\pi D, \quad (5\cdot32)$$

$$Z_{02} = \rho_0\omega/c_0 + 1/D \cdot \left[\{\rho_0\omega/kS - kS\rho_0\omega(k^2/4\pi^2 + 1/c_{0l}^2)\} \frac{\sin 2kl}{2} + \rho_0\omega/c_{0l} \cdot \cos 2kl \right], \quad (5\cdot33)$$

where

$$D = \cos^2 kl + k^2S^2(k^2/4\pi^2 + 1/c_{0l}^2) \sin^2 kl - kS/c_{0l} \cdot \sin 2kl. \quad (5\cdot34)$$

If the tube is closed at the end, we have $Z_l = \infty$. From our expression (5·30) it therefore follows that

$$Z_0 = -i\rho_0 c/S \cdot \cot kl + i\rho_0\omega/c_0, \quad (5\cdot35)$$

whence in this case

$$Z_{01} = 0 \quad \text{and} \quad Z_{02} = \rho_0\omega/c_0 - \rho_0 c/(S \tan kl). \quad (5\cdot36)$$

With the substitution of the above values into the general expression for the power transmission (eq. (5·5) of Sec. 5·1) we get formulae from which calculations may be made on special cases.

As usual, the difficult quantities to estimate theoretically are the conductivities c_0 and c_{0l}. For the conduit end of the tube perhaps the safest formula is

$$c_0 = \pi a^2/(L + \pi a/4), \qquad (5{\cdot}37)$$

where L is here the length of the orifice channel. In this case L is so small compared with a as to be usually quite negligible. Hence practically $c_0 = 4a$. As regards the open end our only ultimate resource is the actual experimental values. Thus from Barton, Textbook of Sound (p. 251), we take $c_{0l} = 5{\cdot}5a$.

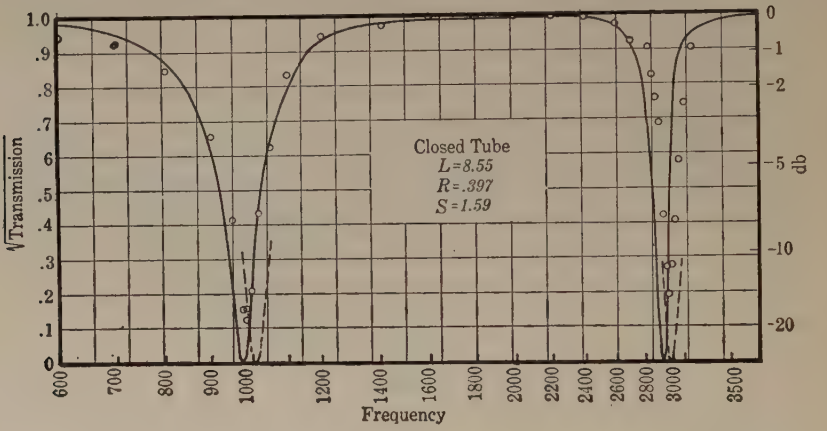

Fig. 5·8.

The comparison between the calculated and experimental results is shown for two special cases [1] in Figs. 5·8 and 5·9. The first of these refers to a closed tube 8.55 cm long with $a = 0.397$ cm, the area of cross section of the conduit being 1.59 cm^2 as in the experiments described in the preceding section. The ordinates are the square roots of the transmission as calculated by the eq. (5·5) of Section 5·1. The circles represent the experimentally determined values. There is a two-fold difference. First, the experimental points do not come down to zero at the resonance frequencies. This is due to the viscous resistance already discussed in connection with the Helmholtz resonator (Sec. 5·2). Second, the computed points are displaced toward smaller frequencies, which may be accounted for by the fact that the value chosen for c_0 ($= 4a$) is too small. The

[1] G. W. Stewart, Phys. Rev., **27**, 494, 1926.

dotted minimum curves are computed for $c_0 = \infty$, and the correct curve should fall somewhere in between. The reason that we ought not to expect the value $c_0 = 4a$ to give the right result is that this is evaluated on the basis of a tube open to the air, while we are applying it to the case of a tube opening into a conduit. The increase in the resonance transmission with increasing frequency, which is very marked in the experimental points, is undoubtedly due to viscosity, the effect of which increases markedly with the frequency.

In the second figure (Fig. 5·9) are presented the results of an investigation of an *open* tube 17·1 cm long with the remaining data as in the previous example. Due to the greater length the viscosity

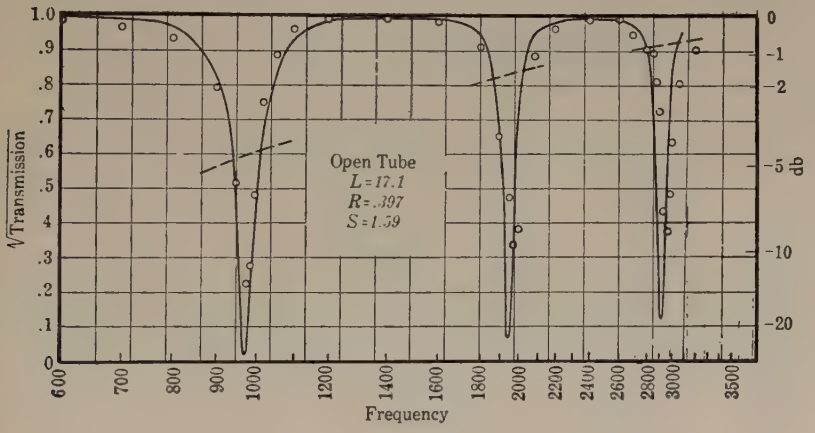

Fig. 5·9.

effect should be larger and the presence of radiation from the open end will work in the same direction. This is what the curves indicate. The dotted line shows the transmission that would be expected in the conduit if $l = 0$ and we had merely an orifice instead of the resonating tube. This makes clear the essential difference between the two cases.

5·5. Acoustic Radiation from an Open Pipe.—This is an appropriate point for the discussion of the general correction to the length of an open pipe. In our elementary discussion of tubes in Section 2·5 it was assumed that at the open end of a pipe the excess pressure remains constantly zero, that is, that we have there a *loop*. This is never exactly true because of the radiation of sound energy from

the opening. The presence of this radiation involves the existence of excess pressure, which therefore means that there can really be *no* loop at the end. It is our present problem to discuss the amount of this radiation. Consider the pipe in Fig. 5·10 with diameter $2a$

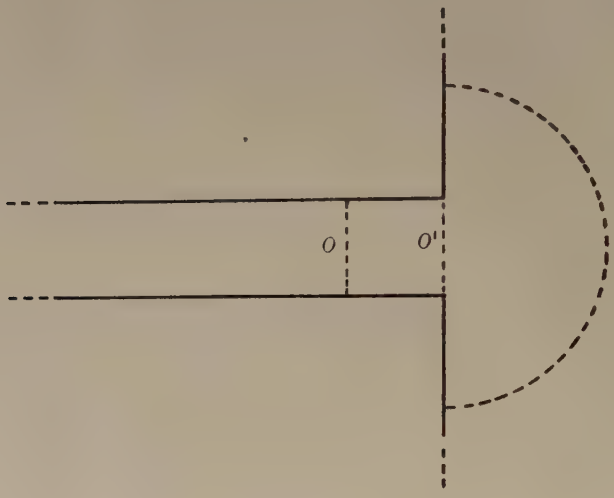

Fig. 5·10.

small compared with the wave length and with the open end fitted with a theoretically infinite flange to insure radiation only in the right hemisphere. The velocity potential at a point in the hemisphere to the right of the flange will be, if we consider O' as a point source of spherical waves,

$$\varphi_e = A'/r \cdot e^{ik(ct-r)}, \tag{5·38}$$

as we note by consulting eq. (1·35) of Section 1·14. It therefore follows that the particle velocity is

$$\frac{\partial \varphi_e}{\partial r} = -A'/r^2 \cdot e^{ik(ct-r)} \cdot (1 + ikr). \tag{5·39}$$

Let us now consider the mass of gas in the region bounded by the plane section through O (to the left of which the waves are plane), and a hemispherical surface with center at O' and radius r', where $2a < r' < < \lambda$. Within this space, which is much less than λ in extent, the gas will move more or less as an incompressible fluid.

The volume current X across the hemispherical surface will be

$$2\pi r'^2 \frac{\partial \varphi_e}{\partial r}\Big)_{r=r'} = -2\pi A'(1 + ikr')e^{ik(ct-r')}$$
$$= -2\pi A' e^{ikct} \qquad (5{\cdot}40)$$

if we neglect terms of the order of $k^2r'^2$ and higher. Note that this implies that $r'/\lambda << 1/2\pi$. In the inside of the pipe the velocity potential will consist of two terms, one for waves in the forward direction and the other for waves in the backward direction. We may therefore write

$$\varphi_i = Ae^{ik(ct-x)} + Be^{ik(ct+x)}. \qquad (5{\cdot}41)$$

The volume current across the section at o, if we let $x = 0$ there, is

$$S\frac{\partial \varphi_i}{\partial x}\Big)_0 = -iSke^{ikct}(A - B), \qquad (5{\cdot}42)$$

where $S = \pi a^2$. From the assumption that the gas in the region under consideration moves as a whole we therefore have

$$iSk(A - B) = 2\pi A'. \qquad (5{\cdot}43)$$

But the volume current through this region is proportional to the difference between the velocity potentials at the two ends, and the coefficient of proportionality is the conductivity of the channel (see Rayleigh, Vol. II, pp. 172–173). Hence

$$-2\pi A' e^{ikct} = c_0(\varphi_e - \varphi_i), \qquad (5{\cdot}44)$$

where φ_e is the value for $r = r'$ and φ_i is the value for $x = 0$. That is, making the substitution,

$$2\pi A' = c_0(A + B - A'/r' \cdot e^{-ikr'}). \qquad (5{\cdot}45)$$

Now eliminate A' between the eqs. (5·43) and (5·45) and we have on reduction, neglecting $c_0/2\pi r'$ compared with unity,

$$B = -A\frac{[c_0^2(1 - S^2k^2/4\pi^2) - S^2k^2 - 2iSkc_0]}{[c_0^2(1 + Sk^2/2\pi)^2 + S^2k^2]}. \qquad (5{\cdot}46)$$

The ratio of the reflected intensity to the incident intensity is given by $|B|^2/|A|^2$ (see Sec. 3·6), and the ratio of the dissipated or radiated intensity to the incident at the end of the tube is

$1 - |B|^2/|A|^2$. Hence the dissipation ratio, as we may call it, is

$$D_r = 1 - |B|^2/|A|^2 = \frac{Sk^2/2\pi}{(1/2 + Sk^2/4\pi)^2 + S^2k^2/4c_0^2} \quad (5\cdot47)$$

after some reduction. This can be put into somewhat more convenient form if we substitute for k its equivalent, $2\pi/\lambda$, and $S = \pi a^2$, and also assume $c_0 = 2a$. Then

$$D_r = \frac{2\pi^2a^2/\lambda^2}{(1/2 + \pi^2a^2/\lambda^2)^2 + \pi^4a^2/4\lambda^2}. \quad (5\cdot48)$$

But since $a < < \lambda$, the formula reduces for all practical purposes to

$$D_r = 8\pi^2a^2/\lambda^2. \quad (5\cdot49)$$

Thus if $a/\lambda = 2/100$ we get $D_r = 3.2\%$, while if $a/\lambda = 4/100$ we get $D_r = 12.8\%$. We must be careful to apply neither eq. (5·48) nor eq. (5·49) to cases where a is too large a fraction of λ. For under the latter conditions the reasoning leading to formula (5·47) breaks down.

It is interesting to note that if the flange be removed we get less dissipation. We may get the approximate value of the dissipation without the flange if we neglect the influence of the flange on the value of c_0. For in the absence of the flange, the flow from O' will be nearly spherical and the volume current will then be approximately

$$-4\pi A' e^{ikct}$$

instead of $-2\pi A' e^{ikct}$ as formerly. If we carry this factor of 2 through the above reasoning, we arrive finally at the approximate value

$$D_r = 4\pi^2a^2/\lambda^2 \quad (5\cdot50)$$

corresponding to $8\pi^2a^2/\lambda^2$ under the same conditions except with the flange present. This conclusion at first seems strange. Surely the opening to the free air would permit the wave to escape more easily than when a barrier (a flange) is imposed! But it is the increase of the area that causes reflection. A continued tube would give complete transmission at this point. Even in a case not within the limits of the assumptions the above results indicate how small an amount of sound energy is dissipated from the open end of a pipe the radius of which is small compared with the wave length. Incidentally by the application of the Helmholtz reciprocal theorem

(see Sec. 1·11) the fact may be established that if we have a source of sound at a point outside a pipe satisfying the above conditions, only as much of the sound energy from the source will get into the pipe as would come out of the pipe and reach that particular point if the same source were in the pipe. Thus in general a small opening is very inefficient either for the entrance or egress of sound. The foregoing discussion shows that as a is decreased in size the dissipation is relatively less. Assuming that we remain within the limits of our assumption that a/λ is very small, it is easily seen that an increase in a increases the radiation and that if one desired to get radiation out of a pipe one should increase the diameter of the opening. But if this is done suddenly a reflection ensues. The result of such qualitative considerations would lead one to build a flaring or bell-like end if radiation is desired. The transmission from inside out or vice versa is thereby increased. Used as a receiver the funnel-shaped end acts not so much to collect the sound as to supply easier ingress.

Questions and Problems

1. Discuss from a physical point of view the influence of an orifice as a branch and in particular explain why the transmission through a conduit with this kind of branch should increase with the frequency.

2. Estimate from physical considerations the effect of (*a*) a pinhole orifice, (*b*) a very narrow short cylindrical tube. Using the material of Sections 5·1 and 5·3, carry through the calculation in the case (*b*) to find the transmission.

3. What are two objects in having flaring ends for speaking tubes?

CHAPTER VI

Distributed Acoustic Impedance.—Horn Theory.—Acoustic Coupling

6·1. Impedance Theory of Tubes and Horns.—Let the accompanying figure (Fig. 6·1) represent a tube or conduit of varying cross section and with diameter *small* compared with the wave length of the sound supposed to be passing through it. From this assumption it follows that the phase will remain approximately constant over every plane perpendicular to the axis. Consider a

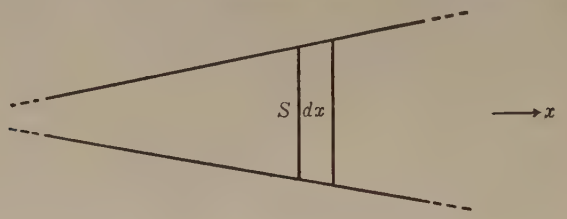

Fig. 6·1.

thin lamina of thickness dx and area of cross section S. Its mass is $\rho_0 S dx$ if the mean density is ρ_0. The excess pressure at the left hand boundary is p; that at the right is $p + (\partial p/\partial x)dx$, Whence the net excess pressure in the x direction is $-(\partial p/\partial x)dx$, corresponding to a force of $-S(\partial p/\partial x)dx$. The equation of motion of the layer is then given by

$$\rho_0 S \ddot{\xi} dx = -S \frac{\partial p}{dx} \partial x$$

or

$$\rho_0 \ddot{\xi} = -\frac{\partial p}{\partial x} \tag{6·1}$$

if we denote the displacement in the x direction by ξ, as usual. It is desirable, however, to get an equation involving p alone and another involving ξ alone. To do this it is necessary to employ the equation of continuity (Sec. 1·12 and eq. (1·9)) according to which the difference between the flow into the lamina and the flow out of

it per second is equal to the time rate of increase in mass of the fluid inside. The amount of the influx over the efflux is

$$-\frac{\partial}{\partial x}(\rho S\dot{\xi})dx,$$

while the rate of increase of mass of fluid is

$$\frac{d}{dt}(\rho S dx) = \dot{\rho} S dx.$$

Whence the equation of continuity may be written

$$-\frac{\partial}{\partial x}(\rho S\dot{\xi}) = S\dot{\rho}. \tag{6·2}$$

Now from the definition of the condensation s (see Sec. 1·12), we have

$$\dot{\rho} = \rho_0 \dot{s} = \frac{1}{c^2}\dot{p} \tag{6·3}$$

since $p = \rho_0 c^2 s$ (eq. (1·14)). Therefore we get

$$-\frac{\partial}{\partial x}(\rho S\dot{\xi}) = \frac{S}{c^2}\dot{p}. \tag{6·4}$$

Expanding and substituting $\rho = \rho_0(1 + s)$ by the definition of condensation, we have, neglecting $\rho_0 S\dot{\xi}(\partial s/\partial x)$ as compared with $\rho_0 S(\partial\dot{\xi}/\partial x)$ and $\dot{\xi}\rho_0 s(\partial S/\partial x)$ as compared with $\rho_0\dot{\xi}(\partial S/\partial x)$,

$$\rho_0\left[S\frac{\partial\dot{\xi}}{\partial x} + \dot{\xi}\frac{\partial S}{\partial x}\right] = -\frac{S}{c^2}\dot{p}. \tag{6·5}$$

Differentiation with respect to the time yields

$$\rho_0\frac{\partial}{\partial x}(S\ddot{\xi}) = -\frac{S}{c^2}\ddot{p}. \tag{6·6}$$

But from the equation of motion we have

$$\ddot{\xi} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x},$$

whence there follows on substitution

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{S}\cdot\frac{\partial S}{\partial x}\cdot\frac{\partial p}{\partial x} = \frac{\ddot{p}}{c^2}. \tag{6·7}$$

If p is a harmonic function of the time with frequency $\omega/2\pi = kc/2\pi$, i.e., the tube is excited by a harmonic vibration of this frequency, the above equation becomes

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial(\log S)}{\partial x} \cdot \frac{\partial p}{\partial x} + k^2 p = 0. \tag{6·8}$$

We can get a corresponding equation in ξ by looking back to eq. (6·5), integrating with respect to the time (putting the constant of integration equal to zero by the appropriate choice of time origin), and differentiating with respect to x. But this step is unnecessary for our present purpose.

We may summarize briefly the fundamental assumptions which have led to the eq. (6·8).[1] They are as follows:

I. The cross-sectional area $S << \lambda^2$.

II. The fundamental acoustic equations are applicable.

III. Both ξ and p are everywhere analytic functions of time and space so that we can interchange the x and t derivatives.

IV. The walls of the tube are rigid; there is no displacement in a direction perpendicular to the axis.

V. $\dot{\xi}$ and s are both so small that we can neglect $S\dot{\xi}(\partial s/\partial x)$ compared with $S(\partial\dot{\xi}/\partial x)$ and $\rho_0 s$ compared with ρ_0.

It is not to be supposed that all these assumptions are realized in practice. The point is that only by using them is a reasonably simple solution possible. Moreover, the results of the theory are sufficiently in accord with the observed behavior of horns to cast considerable light on these phenomena.[2]

The complete solution of eq. (6·8) will be found in Appendix III. It suffices here to note the result, which is that the pressures and volume displacements at the two ends of the horn (indicated by subscripts 1 and 2 respectively) are related by the equations

$$p_2 = ap_1 + bX_1, \tag{6·9}$$

$$X_2 = fp_1 + gX_1, \tag{6·10}$$

[1] In connection with the use of the equation of motion (6·1) it might at first be thought more correct to write (using force instead of pressure) $\rho_0 S\ddot{\xi} = -\partial(Sp)/\partial x$, which yields a result differing from (6·1) by the term $-p/S \cdot \partial S/\partial x$. In general, the absolute value of this term will be small compared with $|\partial p/\partial x|$. But in any case this "extra" pressure really acts on the walls of the tube and hence gives rise to transverse motions. Our fundamental approximations have neglected these motions. Hence we should be inconsistent in introducing this term without at the same time revising completely the original assumptions to take account of the components $\dot{\eta}$ and $\dot{\zeta}$. For this reason we fail to agree with the method of M. O'Day (Phys. Rev., **32**, 328, 1928).

[2] The above theory is that of A. G. Webster, Proc. Nat. Acad. Sci., **5**, 275, 1919.

where a, b, f, g are functions of the x values for the two ends of the horn and the areas of cross section at the two ends. Exact general formulae for these quantities, whose special values depend on the kind of horn used, are to be found in the Appendix just quoted.

Let the terminal impedances be Z_1 and Z_2. That is, let

$$Z_1 = \frac{p_1}{\dot{X}_1}, \qquad Z_2 = \frac{p_2}{\dot{X}_2}. \tag{6·11}$$

Now Z_1 can be expressed in terms of Z_2 or vice versa. For if we solve (6·9) and (6·10), we get

$$Z_1 = \frac{p_1}{\dot{X}_1} = \frac{p_1}{i\omega X_1} = \frac{1}{i\omega}\left[\frac{gp_2/X_2 - b}{-fp_2/X_2 + a}\right]. \tag{6·12}$$

But since $p_2/X_2 = i\omega Z_2$, the above becomes

$$Z_1 = \frac{i\omega g Z_2 - b}{i\omega a + \omega^2 f Z_2}. \tag{6·13}$$

Similarly we obtain

$$Z_2 = \frac{i\omega a Z_1 + b}{i\omega g - \omega^2 f Z_1}. \tag{6·14}$$

Whence if we know the impedance at one end of the horn, we can calculate that at the other as soon as we know a, b, f, g. We are now ready to apply the results of this section to the important problem of the amplification of a horn.

6·2. Amplification of a Horn.—Consider the horn indicated in Fig. 6·2. Its shape is arbitrary and it is represented as a cone merely for the sake of simplicity. The theory of the preceding section tells us that if we know the pressure and volume displacement in the horn at either end, we can compute by (6·9) and (6·10) the corresponding quantities at the other. For example, if the horn is excited at the small end (denoted by 1), we can calculate p_2 and X_2 at the large end and from these in turn can obtain the energy radiation from the opening if we treat the latter as a simple orifice with impedance

FIG. 6·2.

$$Z_0 = \frac{\rho_0 \omega k}{2\pi} + \frac{i\rho_0 \omega}{c_0},$$

which is, of course, an approximation, implying the presence of a flange on the opening; see Section 5·3 and eq. (5·31) of Section 5·4. In the present section, however, we wish to consider the horn as a receiver of sound originating at a distance and approaching the large end. When a steady state is established, there will be at the end 2 an excess pressure p_3 due to the *incident* sound alone (i.e., the pressure which the incident sound would produce at this point if the horn were absent) and an excess pressure p_2 due to the fact that the horn is present and is excited by the incident radiation. The former pressure corresponds to a volume current *towards* the opening, while the latter corresponds to one *out* of the opening. The net volume current *out* will then be

$$\dot{X}_2 = \frac{p_2}{Z_0} - \frac{p_3}{Z_0} = \frac{p_2 - p_3}{Z_0}. \tag{6·15}$$

In turn there will be produced at the small end of the horn an excess pressure p_1, depending on the value of X_2 above, and we may take as a measure of the amplifying power of the horn the ratio p_1/p_3. The calculation of this quantity is as follows: Noting [1] that $p_2/\dot{X}_2 \doteq Z_2$, we have

$$\dot{X}_2 = \frac{Z_2 \dot{X}_2 - p_3}{Z_0} = \frac{p_3}{Z_2 - Z_0}. \tag{6·16}$$

But from the general theory in Section 6·1 (eq. (6·10)) we also have

$$X_2 = f p_1 + g X_1$$

or

$$\dot{X}_2 = i\omega f p_1 + g \frac{p_1}{Z_1}, \tag{6·17}$$

[1] The reader must be careful not to confuse Z_2 and Z_0. The former is simply the general expression for the impedance at the end 2 and its value will depend on the behavior of the horn (i.e., whether emitting or receiving) and on the attachment to this end, while the latter refers strictly to the end considered as an orifice. In the present case as $p_3 \doteq 0$, $Z_2 \doteq Z_0$. It should be pointed out that Webster, to whom the above theory in its original form is due, considered Z_0 as a two-point impedance rather than the usual point impedance of acoustics. As a matter of fact it is given the latter significance in all applications of the resulting formulae and hence the theory has here been developed accordingly.

where $Z_1 = p_1/\dot{X}_1$ as in Section 6·1.[1] We may therefore solve for p_1 and get

$$p_1 = \frac{\dot{X}_2}{\left(i\omega f + \frac{g}{Z_1}\right)} = \frac{Z_1 p_3}{(Z_2 - Z_0)(i\omega f Z_1 + g)}. \qquad (6\cdot18)$$

We should like to eliminate the Z_2 from the above expression. This may be done by employing eq. (6·14) of Section 6·1, whence we finally obtain on substitution into eq. (6·18)

$$p_1/p_3 = \frac{i\omega Z_1}{(i\omega a Z_1 + b) - Z_0(i\omega g - \omega^2 f Z_1)} \qquad (6\cdot19)$$

in terms of the characteristics of the horn (as given by a, b, f and g) and terminal impedances Z_0 and Z_1. The ratio p_1/p_3 may be called the pressure amplification of the horn. The intensity amplification is more important, and will be given by $|p_1|^2/|p_3|^2$ (see Sec. 5·1). Applications of eq. (6·19) will be made in sections which follow.

6·3. The Cylinder and Cone.—The case of a cylindrical tube has already been treated separately in Section 5·4 as a branch to a conduit, but we shall now show how it follows easily from the general theory laid down in the previous section. Here S is constant and eq. (6·8) reduces to the form[2]

$$\frac{\partial^2 p}{\partial x^2} + k^2 p = 0. \qquad (6\cdot20)$$

By the method indicated in Appendix III, we find for the constants of the cylinder

$$\begin{aligned} a &= \cos k(x_2 - x_1) = \cos kl, \\ b &= \frac{\beta}{S}\sin kl, \\ f &= -\frac{S}{\beta}\sin kl, \\ g &= \cos kl, \end{aligned} \qquad (6\cdot21)$$

where $l = x_2 - x_1$ is equal to the length of the tube and $\beta = kc^2\rho_0$.

[1] In the unusual case of end 1 being open also, we should take account of the pressure p_3 there as well. However, even in this case, due to its small relative value, it may be safely neglected.

[2] Since in eq. (6·20) p is considered as a function of x only we might better use total rather than partial differentials. It is believed, however, that the reader will not misunderstand the present notation. The same remark applies to eqs. (6·25) and (6·33).

The impedance then becomes from (6·13)

$$Z_1 = \frac{i\omega \cos kl \cdot Z_2 - \frac{\beta}{S}\sin kl}{-\frac{\omega^2 S}{\beta}\sin kl \cdot Z_2 + i\omega \cos kl}. \tag{6·22}$$

Suppose that the end denoted by x_2 is an orifice. Then (see eq. (5·20) and (5·21) in Sec. 5·3 and note that we here neglect the viscosity effect entirely), we have

$$Z_2 = \frac{\rho_0 \omega k}{2\pi} + \frac{i\rho_0\omega}{c_{0l}}, \tag{6·23}$$

where c_{0l} is the conductivity of the orifice. It is of interest to notice that in our discussion of the cylindrical tube as a side branch we deduced a formula (eq. (5·30)) for the impedance at one end in terms of that at the other end, which is identical with our eq. (6·22) except that we used previously Z_0 and Z_l in place of Z_1 and Z_2 respectively. Of course Section 5·4 is really a special case of our general treatment. Substitution of Z_2 from eq. (6·23) into eq. (6·22) then yields an expression which is identical with the second term in eq. (5·33) (i.e., all except the term $\rho_0\omega/c_0$). We refer to Section 5·4 for the details.

The conical pipe was investigated in Section 3·7. We now examine the application of the general method. We have

$$S = S_0 x^2,$$

whence

$$\frac{\partial \log S}{\partial x} = \frac{2}{x}, \tag{6·24}$$

and the eq. (6·8) becomes

$$\frac{\partial^2 p}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial p}{\partial x} + k^2 p = 0. \tag{6·25}$$

The fundamental constants, calculated in the usual way (see Appendix III) are, placing $kx_1 = \tan k\epsilon_1$ and $kx_2 = \tan k\epsilon_2$ for simplicity,

$$\begin{aligned} a &= \frac{x_1}{x_2}\left(\cos kl + \frac{\sin kl}{kx_1}\right) = \frac{x_1}{x_2}\frac{\sin k(l+\epsilon_1)}{\sin k\epsilon_1}, \\ b &= \frac{\beta x_1}{S_1 x_2}\sin kl, \\ f &= -\frac{S_2 x_1}{\beta x_2}\cdot\frac{\sin k(l+\epsilon_1-\epsilon_2)}{\sin k\epsilon_1 \cdot \sin k\epsilon_2}, \\ g &= -\frac{S_2 x_1}{S_1 x_2}\cdot\frac{\sin k(l-\epsilon_2)}{\sin k\epsilon_2}. \end{aligned} \tag{6·26}$$

Substitution into eq. (6·14) of Section 6·1 yields

$$Z_2 = -\left(\frac{\beta}{S_2}\right) \cdot \frac{i\omega Z_1 \cdot \dfrac{\sin k(l+\epsilon_1)}{\sin k\epsilon_1} + \dfrac{\beta}{S_1} \cdot \sin kl}{-\omega^2 Z_1 \cdot \dfrac{\sin k(l+\epsilon_1-\epsilon_2)}{\sin k\epsilon_1 \cdot \sin k\epsilon_2} + \dfrac{i\omega\beta}{S_1} \cdot \dfrac{\sin k(l-\epsilon_2)}{\sin k\epsilon_2}} \cdot \quad (6\cdot27)$$

If the conical tube is closed at the end corresponding to $x = x_2$, we have $Z_2 = \infty$ and hence

$$Z_1 = \frac{i\beta}{S_1\omega} \cdot \frac{\sin k\epsilon_1 \cdot \sin k(l-\epsilon_2)}{\sin k(l+\epsilon_1-\epsilon_2)}, \quad (6\cdot28)$$

that is, the impedance of the other end is now a pure reactance. On the other hand, if the large end is an orifice, we may set, as before,

$$Z_2 = \frac{\rho_0\omega k}{2\pi} + \frac{i\rho_0\omega}{c_{0l}}.$$

The resulting Z_1 is very complicated, containing both resistance and reactance components, and will not be set down here. In any case we are usually more interested in the amplification produced by the horn as a receiver, from which expression Z_2 has disappeared. Thus we have from eq. (6·19)

$$p_1/p_3 = \frac{Z_1}{aZ_1 + b/i\omega - Z_0(i\omega f Z_1 + g)},$$

wherein the appropriate values of a, b, f, g from eqs. (6·26) must be inserted. The above can be written

$$p_1/p_3 = \frac{1}{a - i\omega f Z_0 + \dfrac{b}{i\omega Z_1} - g \cdot \dfrac{Z_0}{Z_1}}, \quad (6\cdot29)$$

and it appears to be most simple to consider a conical horn with the small end closed, so that $Z_1 = \infty$. Incidentally, we note that if the horn is used to amplify sound for the ear so that the small end is inserted into the ear this assumption is practically realized. Under this condition the amplification reduces to the simple form

$$p_1/p_3 = \frac{1}{a - i\omega f Z_0}$$

$$= \frac{x_2/x_1}{\dfrac{\sin k(l+\epsilon_1)}{\sin k\epsilon_1} + \dfrac{S_2\rho_0\omega^2}{\beta} \cdot \dfrac{\sin k(l+\epsilon_1-\epsilon_2)}{\sin k\epsilon_1 \cdot \sin k\epsilon_2} \cdot \left(\dfrac{ik}{2\pi} - \dfrac{1}{c_{0l}}\right)}, \quad (6\cdot30)$$

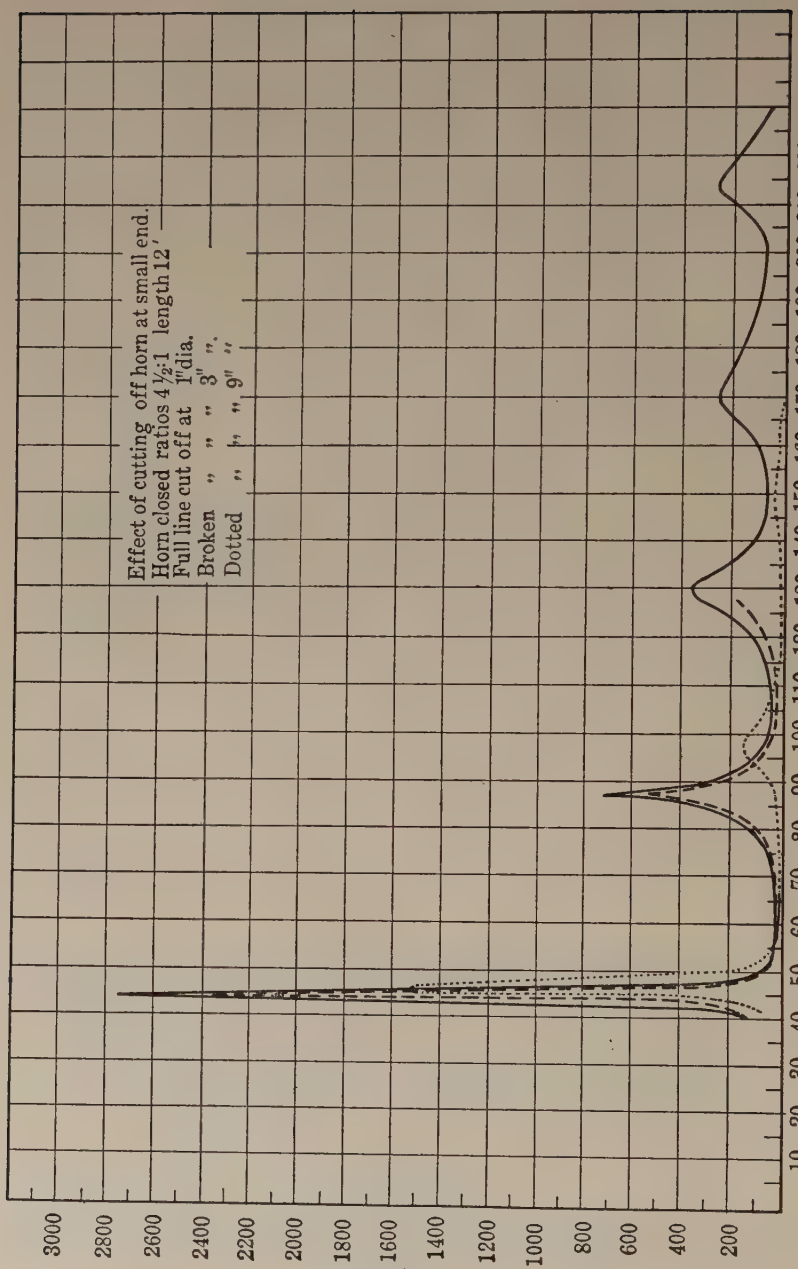
Effect of cutting off horn at small end.
Horn closed ratios 4½:1 length 12'
Full line cut off at 1" dia.
Broken " " " 3" ".
Dotted " " " 9" "
$|p_1|^2/|p_3|^2$
3000
2800
2600
2400
2200
2000
1800
1600
1400
1200
1000
800
600
400
200
10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240
Frequency

FIG. 6·3.

after making the substitutions for a and f and using

$$Z_0 = \frac{\rho_0 \omega k}{2\pi} + \frac{i\rho_0 \omega}{c_{01}}.$$

If the vertex of the horn is closed, we must put $x_1 = 0$ in (6·30), which then takes the form

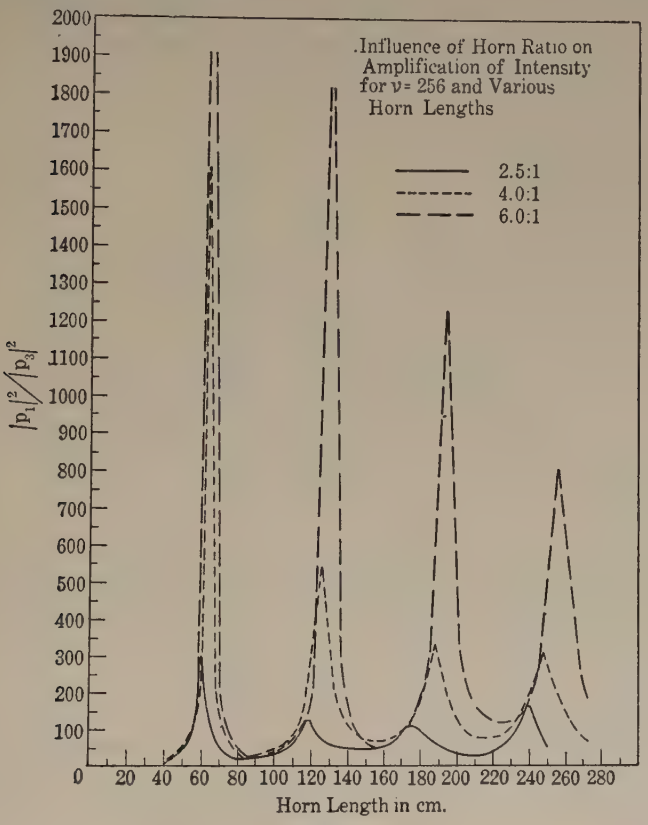
Influence of Horn Ratio on Amplification of Intensity for ν = 256 and Various Horn Lengths
2.5:1
4.0:1
6.0:1
$|p_1|^2/|p_3|^2$
Horn Length in cm.

Fig. 6·4.

$$p_1/p_3 = \frac{1}{\dfrac{\sin kx_2}{kx_2} + \dfrac{S_2\rho_0\omega^2}{\beta} \cdot \dfrac{\sin k(x_2 - \epsilon_2)}{kx_2 \sin k\epsilon_2} \left(\dfrac{ik}{2\pi} - \dfrac{1}{c_{01}} \right)}. \quad (6\cdot31)$$

The influence of the value of x_1 (i.e., the distance of the cut-off from the vertex of the cone of which the horn is a frustrum) on the amplification is shown in the diagram (Fig. 6·3), presenting curves

computed with the use of eq. (6·30). (Note, however, that the ordinates are values of $|p_1|^2/|p_3|^2$.) The increase in the intensity amplification for the resonance frequencies as x_1 is made smaller (i.e., smaller end) is very marked, particularly for the overtones.

The influence of the horn ratio (that is, the ratio between the length of the horn and the diameter at the large end) on the amplification is well illustrated by the above diagram (Fig. 6·4) in

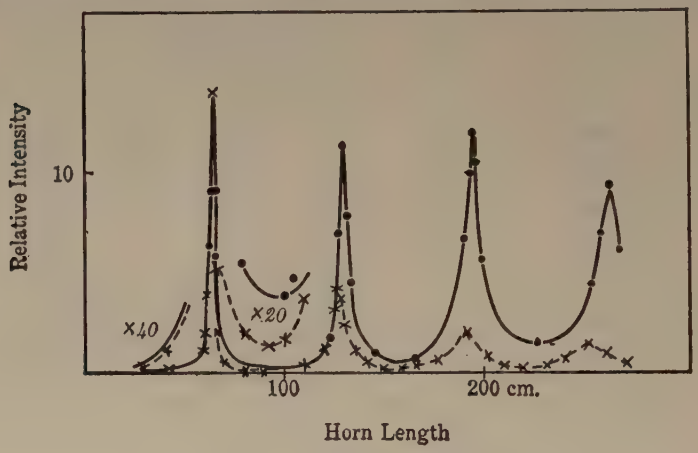

Fig. 6·5.

which $|p_1|^2/|p_3|^2$ is plotted as a function of horn length for three different ratios. As is of course evident from an examination of the eq. (6·30), the horns with greater ratio show a greater amplification throughout almost the whole range of lengths and in particular in the neighborhood of the lengths corresponding to resonance at the given frequency (256 cycles).

Experiments on the performance of conical horns carried out by one of the authors [1] are in tolerably good agreement with the theory as above presented. The accompanying figure (Fig. 6·5) indicates the nature of the agreement between experimental and theoretical results when a conical horn is used as a receiver. The ordinates are values of $|p_1|^2/|p_3|^2$, while the abscissae are horn lengths in centimeters. The full line curve represents experimental results for conical horns of constant horn ratio cut off with a very small vertex (radius 0.25 cm), the intensity measurements being made with a Rayleigh disc (see Sec. 8·4, Chap. VIII). The dotted curve repre-

[1] G. W. Stewart, Phys. Rev., 16, 313, 1920.

sents the theoretical values computed from eq. (6·31), which strictly speaking implies $x_1 = 0$, but should be a very close approximation if x_1 is sufficiently small. The quantitative agreement is not particularly good though the trend of the two curves is the same.

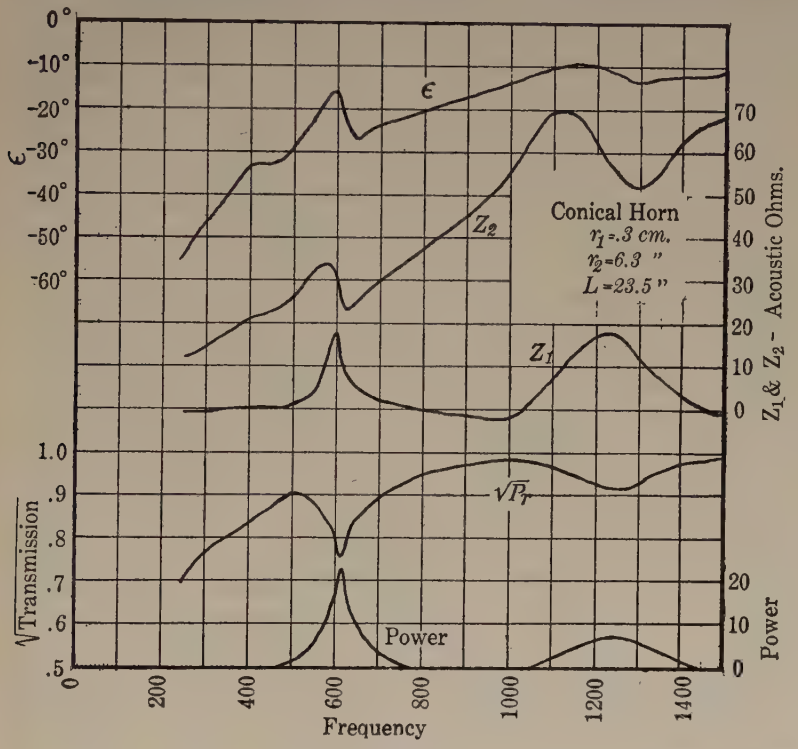

Fig. 6·6.

Reference should be made to the original article for further details, but we ought to call attention here to the important fact indicated by the experiments that actually there exists an optimum horn ratio or horn angle for a horn designed to resonate at a certain frequency. That is, if conical horns of various angles, all constructed to resonate at a given frequency, are used successively as receivers of a given train of sound waves of this frequency, one horn of optimum angle will give the greatest amplification. The optimum angle will depend on whether the fundamental is excited or an overtone. It will be noted that the theory of this section as expressed in eqs. (6·30) and (6·31) indicates no optimum angle but

rather an increase in amplification with an increase in the horn ratio or decrease in horn angle. But in getting (6·30) and (6·31) we assumed the small end to be closed and $Z_1 = \infty$. If this is no longer assumed, but it is postulated that energy dissipation can take place at the small end as well as the large end, and that when a steady state is established the energy input into the horn is equal to the energy output from both ends, a theoretical deduction shows clearly the necessity for the optimum angle. This has been carried through by Hoersch.[1]

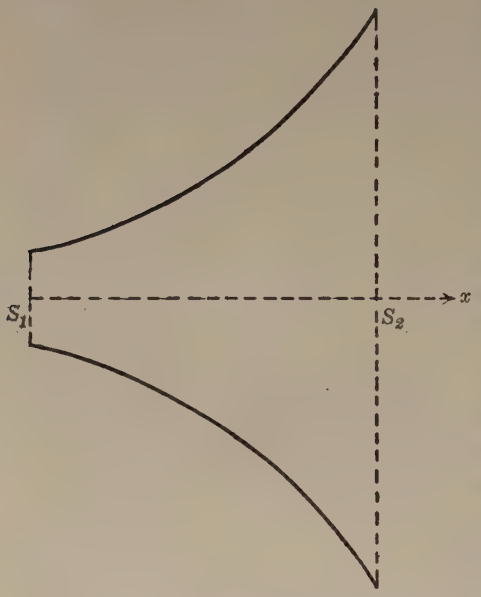

Fig. 6·7.

It will be shown in Section 6·7 that an important measure of the amplification of any horn both as a receiver and as a transmitter is to be found in Z_{01}, the *real* component of the impedance at the small end or throat of the horn. In this connection the accompanying figure (Fig. 6·6), showing the result of measuring both the real and imaginary impedance components at the throat of the horn, is interesting. The method by which the measurements were made is described in Section 8·2.

We should note here the variation of Z_{01} (in the figure denoted by Z_1 simply) with frequency, and the fact that it attains its maximum value slightly in advance of the resonance frequency in each case. The reader should compare this curve with the corresponding ones for exponential and hyperbolic horns of similar dimensions in Sections 6·4 and 6·5.

6·4. The Exponential Horn.—The exponential horn is one for which the cross sectional area is governed by the equation (see Fig. 6·7)

$$S = S_0 e^{mx}. \tag{6·32}$$

[1] V. A. Hoersch, Phys. Rev., **25**, 225, 1925.

Then since $\log S = \log S_0 + mx$, the pressure equation becomes

$$\frac{\partial^2 p}{\partial x^2} + m\frac{\partial p}{\partial x} + k^2 p = 0. \tag{6·33}$$

If $\gamma = \sqrt{k^2 - m^2/4}$ and $l = x_2 - x_1$, we have in the usual way (see Appendix III) for the constants of the horn

$$\begin{aligned} a &= e^{-(ml/2)}\left[\frac{m}{2\gamma}\sin\gamma l + \cos\gamma l\right], \\ b &= \frac{\beta k}{S_1\gamma}e^{-(ml/2)}\sin\gamma l, \\ f &= -\frac{S_2 k}{\beta\gamma}e^{-(ml/2)}\sin\gamma l, \\ g &= \frac{S_2}{S_1}e^{-(ml/2)}\left[-\frac{m}{2\gamma}\sin\gamma l + \cos\gamma l\right]. \end{aligned} \tag{6·34}$$

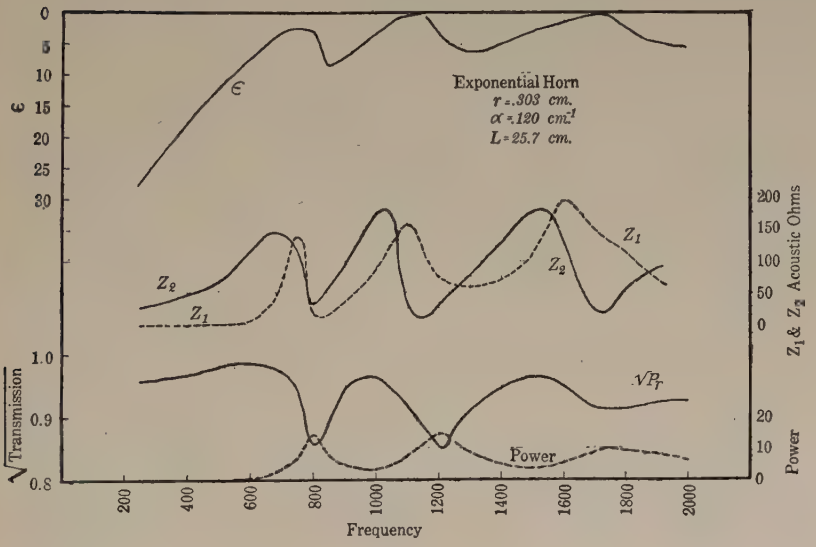

FIG. 6·8.

Substituting into the pressure amplification expression eq. (6·19) of Section 6·2 and considering the small end closed, so that $Z_1 = \infty$, we deduce

$$\begin{aligned} p_1/p_3 &= \frac{1}{a - i\omega f Z_0} \\ &= \frac{\sqrt{S_2/S_1}}{\dfrac{m}{2\gamma}\sin\gamma l + \cos\gamma l + \dfrac{S_2 k^2}{\gamma}\left(\dfrac{ik}{2\pi} - \dfrac{1}{c_0 l}\right)\sin\gamma l}. \end{aligned} \tag{6·35}$$

There is a peculiarity about the exponential horn which has given rise to considerable discussion. The general solution of the differential equation (6·33) is

$$p = e^{-(mx/2)}[Ae^{i(\omega t+\gamma x)} + Be^{i(\omega t-\gamma x)}], \qquad (6{\cdot}36)$$

where $\gamma = \sqrt{k^2 - m^2/4}$ as above. That is, there are two pressure waves in opposite directions, each travelling with the phase velocity

$$2\pi\nu/\gamma = \frac{kc}{\sqrt{k^2 - m^2/4}},$$

where $k = \omega/c$ as usual. Now as $k \doteq m/2$, the phase velocity approaches *infinity*, and for $k \leqq m/2$ there is no imaginary term involving x, the phase then becomes and remains the same at all parts of the horn and there is no longer *any* wave motion at all in the horn, a conclusion startlingly at variance with the fact that we know that the horn will actually transmit low frequencies. Moreover, according to the theory we are using it is only a horn of *exactly* the exponential type which shows this peculiarity. It is difficult to believe that in practice the solution should be so sensitive to slight changes in shape as occur for example in going from a Bessel horn [1] for which $S = S_0x^m$ to the exponential horn, which is the limiting case of that type as m grows very large, when we consider that no such discontinuity arises for the former type of horn. We are therefore forced to conclude that the nature of the approximations underlying Webster's theory here introduces a spurious result. It is probable that a more accurate horn theory would indicate the presence of wave motion in the exponential horn for all frequencies.

The exponential horn of infinite length has been studied by Hanna and Slepian,[2] while Ballantine (loc. cit.) (as has already been noted) has investigated mathematically infinite horns with the cross section equations $S = S_0x^m$, the so-called Bessel horns, the shape of which in the limit of very large m approaches that of the exponential horn. The value of this work seems somewhat limited in view of the great influence of reflection in the finite horns actually in use. Mention should here be made of the article on loud speaker horns by Goldsmith and Minton.[3] This contains much in the way of

[1] Stuart Ballantine, Jour. of the Franklin Institute (**203**, 85, 1927).

[2] Hanna and Slepian, Trans. A. I. E. E., **43**, 393, 1924.

[3] A. N. Goldsmith and J. P. Minton, Inst. Rad. Eng., **12**, 423, 1924.

experimental data and theory which is essentially equivalent to that presented in this text.

The figure above (Fig. 6·8) presents interesting experimental evidence of the superiority of the exponential horn over the conical horn in the measured values of Z_{01} (called Z_1 in the figure), the real component of the impedance at the throat of the horn. These measurements have already been referred to at the end of Section 6·3 and a curve presented there for the case of the conical horn The corresponding curve for an exponential horn of similar dimen-

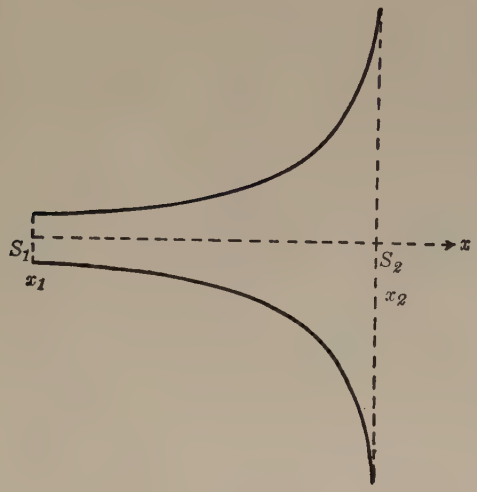

FIG. 6·9.

sions shows clearly larger values of Z_{01}, indicating greater amplifying power, as will be shown in Section 6·7.

6·5. The Hyperbolic and Parabolic Horns.—The hyperbolic horn shown in diagrammatic form in Fig. 6·9 has its cross-sectional area governed by the equation

$$S = S_0 x^{-2}, \tag{6·37}$$

being a special case of the more general form $S = S_0 x^n$. The general pressure eq. (6·8) then takes the form

$$\frac{d^2p}{dx^2} - \frac{2}{x} \cdot \frac{dp}{dx} + k^2 p = 0. \tag{6·38}$$

Proceeding in the usual way, we find for the constants of the horn

$$a = \frac{\sin kl}{kx_1} + \frac{x_2}{x_1}\cos kl,$$

$$b = -\frac{\beta}{k^2 S_0}[\sin kl - kl\cos kl + k^2 x_1 x_2 \sin kl], \tag{6·39}$$

$$f = \frac{S_0}{\beta x_1 x_2}\sin kl,$$

$$g = \frac{kx_1\cos kl - \sin kl}{kx_2}.$$

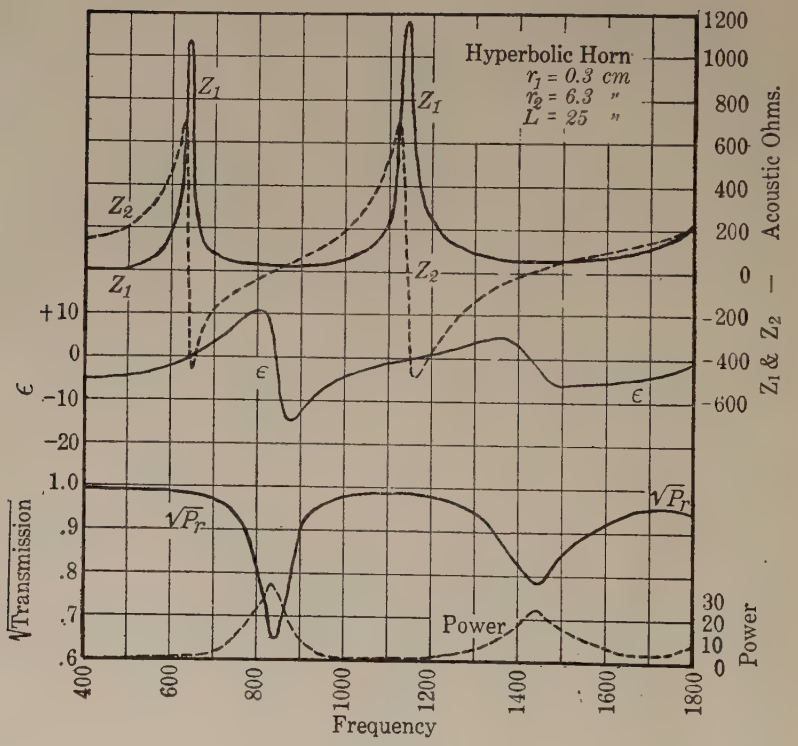

Fig. 6·10.

The pressure amplification calculated on the assumption of $Z_1 = \infty$ and

$$Z_0 = \frac{\rho_0 \omega k}{2\pi} + \frac{i\rho_0\omega}{c_{0l}}$$

turns out to be

$$p_1/p_3 = \frac{1}{-\frac{\sin kl}{kx_1} + \frac{x_2}{x_1}\cos kl + \frac{kS_0 \sin kl}{x_1 x_2} \cdot \left(\frac{ik}{2\pi} - \frac{1}{c_{01}}\right)}. \quad (6{\cdot}40)$$

Computations made with this equation as a basis indicate that the hyperbolic horn is superior to the conical horn of similar dimensions. However, it should be emphasized here that experiment does not indicate much choice between flaring horns. As long as there *is* a flare, the exact law of change of cross section does not seem to be particularly important. This is doubtless connected with the approximate nature of the horn theory used, which perhaps tends to put undue theoretical emphasis on certain factors.

It is of some interest, however, to consider the measurement of the real impedance component at the throat of a hyperbolic horn as indicated in the figure above (Fig. 6·10). This should be compared with the corresponding curves for the conical and exponential horns (Figs. 6·6 and 6·8). The Z_{01} values afford a measure of the horn efficiency.

Very recently Olson and Wolff[1] have examined on the basis of the theory of this chapter the *parabolic* horn for which

$$S = S_0 x.$$

They have calculated p_1/p_3 as above and have found very good agreement with the values obtained experimentally by means of a condenser microphone (see Sec. 8·7).

6·6. Conductivity as a Correction to the Length of a Tube.— In Section 5·5 of the previous chapter there was discussed the radiation of sound from an open pipe, and the influence of this on the effective length of the pipe was mentioned. We shall now examine this in detail, using the general methods of this chapter. We shall use a cylindrical tube open at one end and closed at the other. If the closed end is at $x = x_1$, we have $Z_1 = \infty$, and therefore eq. (6·19) of Section 6·2 becomes

$$p_1/p_3 = \frac{1}{a - i\omega f Z_0}. \quad (6{\cdot}41)$$

[1] H. F. Olson and I. Wolff, Journal of the Acoustical Society of America, 1, 410, 1930.

Referring to Section 6·3, we have for the case of the cylinder (eq. (6·21))

$$a = \cos kl, \qquad f = -\frac{S}{\beta}\sin kl,$$

whence, making the substitution and putting for Z_0 the expression appropriate for an orifice, we have

$$p_1/p_3 = \frac{1}{\cos kl + \left(\frac{ik^2}{2\pi} - \frac{k}{c_{0l}}\right)\sin kl}, \tag{6·42}$$

S being the cross-sectional area of the tube, l its length and c_{0l} the conductivity of the opening. Now for an ordinary circular orifice, c_{0l} is of the order of $2a$, where a is the radius (see Sec. 2·4). But one of the fundamental assumptions underlying this present chapter is that $2a < < \lambda$. Hence, $2a < < 2\pi/k$ and therefore $k/2\pi < < 1/c_{0l}$, whence $k^2/2\pi$ may be neglected in general in comparison with k/c_{0l}. Therefore, we may drop the imaginary term in (6·42) and say that to a good approximation p_1/p_3 will have a maximum for that frequency for which

$$\cos kl - \frac{kS}{c_{0l}}\sin kl = 0. \tag{6·43}$$

Since kS/c_{0l} (which equals $\pi^2 a/\lambda$ approximately) is very small, we can write the above in the form

$$\cos\frac{kS}{c_{0l}}\cos kl - \sin\frac{kS}{c_{0l}}\sin kl = 0$$

or

$$\cos k\left(l + \frac{S}{c_{0l}}\right) = 0,$$

whence the condition becomes

$$k\left(l + \frac{S}{c_{0l}}\right) = \left(n + \frac{1}{2}\right)\pi,$$

where n is any integer or zero. That is,

$$l + \frac{S}{c_{0l}} = \left(n + \frac{1}{2}\right)\cdot\frac{\lambda}{2}. \tag{6·44}$$

This is the revised form of the resonance condition for a tube closed at

one end and open at the other. As we recall from Section 2·6 the simple uncorrected condition is

$$l = \left(n + \frac{1}{2}\right) \cdot \frac{\lambda}{2} \cdot$$

The quantity S/c_{0l} then represents the *end correction* and $l + S/c_{0l}$ is the true acoustic length of the tube. For certain purposes (see Sec. 2·4) it is valuable to *define* c_{0l} in terms of this end correction. Thus,

$$c_{0l} = \frac{S}{\text{End correction}} \cdot \tag{6·45}$$

This gives us a valuable means of measuring c_{0l} experimentally.

6·7. Application of the Reciprocal Theorem to Receivers and Transmitters.—The reciprocal theorem of Helmholtz has already been stated and applied in Section 1·11. It will be of advantage to apply it to the horn. Briefly, it says that a source of sound at a point A will produce at a point B the same velocity potential both as regards magnitude and phase as would be produced at A were the identical source transferred to B. Now the strength of a source of sound may always be measured by the value of the volume current $\dot{X}$ at the source. Two sources are thus equal if their $\dot{X}$ values[1] are equal both in magnitude and phase. This is the definition of equal sources as used in Helmholtz's theorem. Let there be a source $\dot{X}$ at the vertex of a horn. The average power output at the vertex is

$$P = \tfrac{1}{2} p_{\max} \cdot \dot{X}_{\max} \cos\theta, \tag{6·46}$$

where the substituted values are maximum *real* values of excess pressure and volume current, and θ is the phase difference between the two. Now the maximum velocity potential and consequently the maximum excess pressure at a point distant from the horn will be proportional to $\sqrt{\tfrac{1}{2} p_{\max} \dot{X}_{\max} \cos\theta}$ (see eq. (1·50) of Sec. 1·15). Suppose that there is an equal source $\dot{X}$ at the distant point. Then the maximum pressure at the vertex due to this source will also be proportional to $\sqrt{\tfrac{1}{2} p_{\max} \dot{X}_{\max} \cos\theta}$ by the reciprocal theorem. If the source is constant the maximum excess pressure at the vertex will be proportional to $\sqrt{\tfrac{1}{2}|Z| \cos\theta}$ by the definition of the impedance Z; refer at this point to eqs. (3·14), (3·20), (3·22), and (3·23) of Section

[1] See Rayleigh, Vol. II, para. 294.

3·2 for the verification, noting that $|Z|$ means the absolute value of the impedance at the vertex. But $|Z| \cos\theta = Z_{01}$ (Sec. 3·2), where Z_{01} is the *real* part of the impedance at the vertex. Hence, the maximum excess pressure at the vertex is proportional to $\sqrt{Z_{01}}$ and the intensity of reception is proportional to Z_{01}. But when the same source is at the vertex, the *output* is also proportional to Z_{01} from eq. (6·46). Hence, Z_{01} is a measure of the amplification of the horn, both when used as a transmitter and as a receiver, and we see that a horn which is a good transmitter will also be a good receiver.

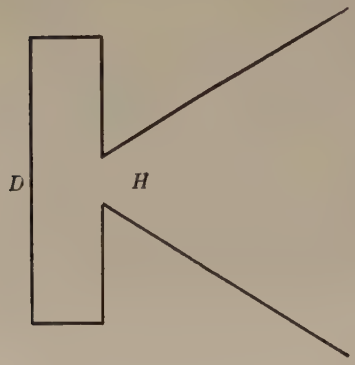

Fig. 6·11.

Naturally, the amplification will depend on the kind of attachment used at the vertex. This will be studied in detail under acoustic coupling in Section 6·8. But we can note here the influence of the most simple kind of attachment, namely, that shown in the figure (Fig. 6·11), in which the diaphragm D is made much larger than the vertex orifice of the horn and is separated from it by a small air space. We know from our study of the stethoscope that sound incident on D may be transmitted with greater power transmission through the small hole than if the hole were not there. We may, therefore, expect that when the diaphragm is made to vibrate a larger $\dot{X}$ will be produced at the opening to the horn than would be produced were the chamber not there. This means a larger output.

It is interesting to note that recently W. Schottky (Zs. f. Phys., **36**, 689, 1926) has made use of the general principle of reciprocity to discuss the emission and reception of sound radiation by sound sources and receivers respectively. In particular, he has proved the following theorem: any sound radiator whose motion is linear and determinable in terms of one coördinate will absorb from unit solid angle of an incident uniform spherical wave of length λ and at distance R from the source an amount of energy which is smaller in the proportion λ^2/R^2 than the amount which it itself is capable of sending per unit solid angle in the direction of the source. That is, while the ratio between the reception and emission efficiencies is the same for any given frequency, this ratio increases with the square of

the wave length, so that low frequencies are *relatively* better received than emitted, i.e., the reception favors the low frequencies. Schottky calls this the law of "depth" reception (Tiefempfang).

6·8. Acoustic Coupling.—Optimum Diaphragm Output.—We have already mentioned in Section 6·7 the possible influence of an orifice attachment on the output of a diaphragm. Such an attachment may be, for example, the cap of a telephone receiver or the horn of a loud speaker. Let us consider the problem in greater detail. We shall not specify at first the exact nature of the attachment. The effective area of cross section of the diaphragm will be denoted by S_1, while the opening into the attachment will have an area of S_2. The thickness of the air layer in front of the diaphragm will be l. This layer will be cylindrical in form. Let Z_1 be the impedance at the diaphragm and Z_2 that at the opening into the attachment. Using now the theory of Section 6·1, we write again (6·22), viz.:

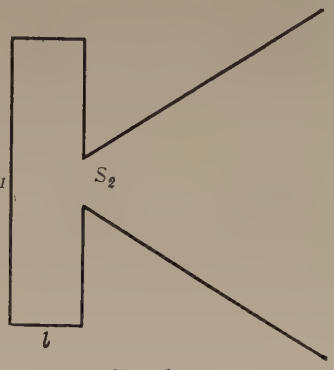

FIG. 6·12.

$$Z_1 = \frac{i\omega Z_2 \cos kl - \frac{\rho_0 kc^2}{S_1} \sin kl}{-\frac{\omega^2 S_1}{\rho_0 kc^2} Z_2 \sin kl + i\omega \cos kl}. \tag{6·47}$$

Now set

$$Z_2 = Z_2' + iZ_2''$$

and

$$Z_1 = Z_1' + iZ_1''.$$

On making the substitution and rationalizing the denominator, we finally have for the real part of Z_1

$$Z_1' = \frac{\rho_0 c}{S_1} \cdot \frac{Z_2'}{D}, \tag{6·48}$$

where

$$D = \frac{S_1}{\rho_0 c}[Z_2'^2 + Z_2''^2] \sin^2 kl + \frac{\rho_0 c}{S_1} \cos^2 kl - Z_2'' \sin 2kl.$$

Now the power output of the diaphragm is

$$P = \tfrac{1}{2} p_{\max} \dot{X}_{\max} \cos\theta = \tfrac{1}{2} \dot{X}^2_{\max} Z_1 \cos\theta$$
$$= \tfrac{1}{2} \dot{X}^2_{\max} Z_1', \qquad (6{\cdot}49)$$

since Z_1' is the real part of Z_1. If the diaphragm were radiating into open air through an opening of the same size as the diaphragm, the power output would be given by

$$\frac{1}{2} R \dot{X}^2_{\max} = \frac{1}{2} \frac{\rho_0 k^2 c}{2\pi} \dot{X}^2_{\max}, \qquad (6{\cdot}50)$$

where R is the dissipation coefficient (see eq. (2·19) in Sec. 2·3), or in this case the radiation resistance of the opening. Hence, the ratio of the output *with* the attachment to that *without* the attachment and no reduction in cross-sectional area, becomes

$$\frac{2\pi Z_1'}{\rho_0 k^2 c} = \frac{2\pi Z_2'}{S_1 k^2 D}, \qquad (6{\cdot}51)$$

on employing eq. (6·50) above.

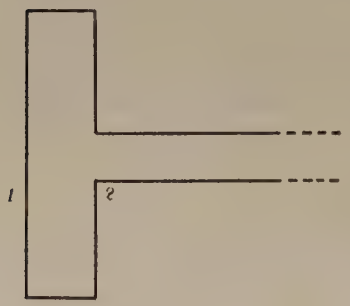

Fig. 6·13.

Coming now to some detailed applications, suppose that the attachment is an infinitely long cylindrical tube, as in the accompanying figure (Fig. 6·13). We have $Z_2'' = 0$, since there is no inertia component at 2 (i.e., no body of air which moves as a whole). Moreover Z_2', the resistance component at 2, is that due to a plane wave at 2. There is no opening in the air at 2, and we are neglecting viscous resistance. Hence

$$Z_2' = \frac{\rho_0 c}{S_2},$$

where S_2 is the area of cross section of the cylindrical tube. Substituting into the above amplification ratio (6·51), we have for the latter

$$\frac{2\pi}{k^2 S_2} \cdot \left[\frac{1}{1 + \sin^2 kl \cdot \left(\frac{S_1^2}{S_2^2} - 1 \right)} \right]. \qquad (6{\cdot}52)$$

The smaller kl, the larger the ratio, and for kl extremely small the ratio varies inversely as S_2. This then gives a measure of the amplifying value of the attachment.

With the orifice open to the air as in Fig. 6·14 we have

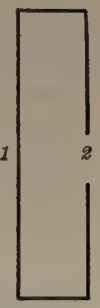

Fig. 6·14.

$$Z_2 = \frac{\rho_0 c k^2}{2\pi} + \frac{i\rho_0\omega}{c_0},$$

where c_0 is the conductivity of the opening. Substitution yields for the amplification ratio

$$\frac{\rho_0 c}{S_1} \cdot \left[\frac{1}{\rho_0 c k^2 S_1 \left(\frac{k^2}{4\pi^2} + \frac{1}{{c_0}^2} \right) \sin^2 kl + \frac{\rho_0 c}{S_1} \cos^2 kl - \frac{2\rho_0\omega}{c_0} \cos kl \cdot \sin kl} \right].$$

If we suppose that l is so small that we can put $\sin kl = kl$ and $\cos kl = 1$, we get

$$\text{Ratio} = \frac{1}{S_1} \cdot \left[\frac{1}{\frac{S_1 k^4 l^2}{{c_0}^2} + \frac{1}{S_1} - \frac{2k^2 l}{c_0}} \right],$$

wherein we have neglected $k^2/4\pi^2$ as compared with $1/{c_0}^2$ (see Sec. 6·6). This ratio will usually be greater than unity, for except for very large values of S_1 we shall have

$$\frac{2k^2 l}{c_0} > \frac{S_1 k^4 l^2}{{c_0}^2},$$

because of the extreme smallness of $k^2 l/c_0$.

Without going into details in the case where the attachment is a horn we can yet understand the amplifying action of the horn when we recall that for a horn Z_2' will be relatively large, as has been indicated in the results of impedance measurements discussed in Sections 6·3, 6·4 and 6·5. Then the amplification ratio for l very small will be practically proportional to Z_2'. (See eq. (6·51) and note that D reduces practically to $\rho_0 c/S_1$.)

It must not be forgotten that the theory of this section is approximate to the extent of assuming that the $\dot{X}_{\max}$ of the diaphragm is not changed by the joining of an attachment such as those described. While this is never true in practice, yet the above

gives a not too complicated picture of the effect of the attachment and is to this extent justifiable.

6·9. Mechanical-Acoustic Coupling.—Illustration.—In practice it is often necessary to couple mechanical and acoustic elements. This necessity arises, for example, in the problem of phonographic reproduction where the mechanical vibrations of a needle point have to produce reasonably undistorted acoustic vibrations in the air. This matter has been very thoroughly worked out by Maxfield and Harrison [1] and it is our aim here to discuss some features of their analysis, in order to supply a concrete case.

The reproducing system is schematically indicated in the following diagram (Fig. 6·15),[2] which shows the needle point and arm ($c_1T_1m_1c_3$), the spider attachment ($c_4m_2c_5$), the corrugated diaphragm (c_6m_3), the air chamber of the horn (c_7T_2) and the throat of the horn (Z_h). With regard to the symbols, m represents effective mass and c mechanical capacitance or "compliance" as the above authors call it. The letter T denotes the coupling of one type of oscillation with another; in the case of T_1 the coupling is mechanical with mechanical, in the case of T_2 the coupling is mechanical with acoustical. The aim is to deduce an equivalent electrical network for every mechanical-acoustical system present. It is not our purpose here to follow this procedure in detail. But it will be valuable to examine briefly a special case and for this purpose we shall choose the air chamber between the diaphragm and the horn. We have the following notation:

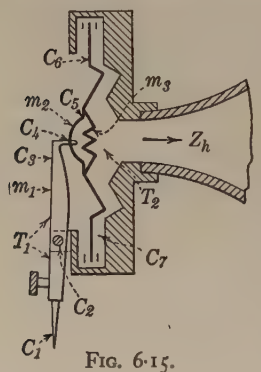

FIG. 6·15.

m_3 = effective mass of diaphragm in grams,
A_1 = equivalent area of diaphragm in cm^2,
c_6 = compliance of the edge of the diaphragm,
c_7 = compliance of air chamber, = acoustic capacitance in our previous notation,
A_2 = area of the throat of the horn,
Z_h = impedance of horn.

[1] J. P. Maxfield and H. C. Harrison, Bell System Technical Journal, **5**, 493, 1926.
[2] Taken from the above article, as is Fig. 6·16.

(Note that the latter is defined by the authors as $\frac{\text{force}}{\text{particle velocity}}$ instead of $\frac{\text{pressure}}{\text{volume current}}$, which has been used throughout this text. To avoid confusion, we shall retain their definition in this section.)

ξ_1 = displacement of diaphragm,
ξ_2 = displacement of air in the horn throat,
F = force applied to diaphragm,
p = excess pressure in the air chamber.

The procedure now is to write down the equation of motion of the diaphragm involving the inertia force, the restoring force due to the compliance of the diaphragm and the restoring force due to the compliance of the air chamber. Next the force equation of the air in the chamber is written down. This is strictly only the equation defining Z_h as above. After some reductions and assuming harmonic displacements, these equations become respectively

$$Z_1\dot{\xi}_1 - Z_m\dot{\xi}_2 = F,$$
$$Z_2\dot{\xi}_2 - Z_m\dot{\xi}_1 = 0,$$

where

$$Z_1 = i\left(\omega m_3 - \frac{1}{\omega c_6} - \frac{1}{\omega c_7}\right),$$
$$Z_2 = \left[Z_h - i\left(\frac{A_2}{A_1}\right)^2 \cdot \frac{1}{\omega c_7}\right],$$
$$Z_m = -i\left(\frac{A_2}{A_1}\right) \cdot \frac{1}{\omega c_7}.$$

Now examination of the following electrical network (see Fig. 6·16) shows that the differential equations giving the sinusoidal currents I_1 and I_2 in terms of the characteristics of the network are of form identical with the acoustic equations above for the particle velocities. In fact we can write, if E denotes the impressed E.M.F.,

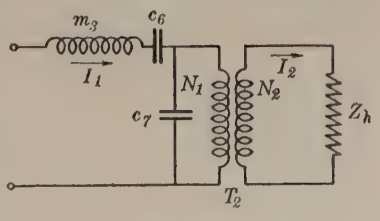

FIG. 6·16.

$$Z_1I_1 - Z_mI_2 = E,$$
$$Z_2I_2 - Z_mI_1 = 0,$$

where

$$Z_1 = i\left(\omega L_3 - \frac{1}{\omega C_6} - \frac{1}{\omega C_7}\right),$$

$$Z_2 = \left[Z_h - i\left(\frac{N_2}{N_1}\right)^2 \cdot \frac{1}{\omega C_7} \right],$$

$$Z_m = -i\left(\frac{N_2}{N_1}\right) \cdot \frac{1}{\omega C_7}.$$

These five equations give the complete solution of the electrical network and hence the complete solution of the analogous mechanical-acoustical problem. Mathematically there is nothing to choose between the two sets of equations. But the fact that the theory of electrical networks has been so extensively studied and is now very familiar to electrical engineers, renders the use of the electrical equations rather more efficient in practical computation. It ought to be emphasized very strongly, however, that this analogy of mechanical-acoustical and electrical networks is valid only to the extent that the corresponding differential equations are similar. Our main point here has been to show how in problems of acoustic coupling, as in problems of electrical coupling, the main thing is to transform force equations into impedance equations, the interpretation of which then becomes immediately clear.

It is well to suggest that the apparent practical gain in the use of analogous electrical equations may prove a detriment to the most rapid progress in anyone's appreciation of acoustic phenomena. The corresponding electrical phenomena are in reality much less concrete. Acoustics deals with gross matter and can be visualized more readily, and the physical action in acoustics is much better understood. Therefore an acoustician might profitably endeavor to think in acoustic terms rather than electrical. The electrical analogies may sometimes be advantageous in mathematical procedure but, in the long run, not in physical interpretation.

Questions and Problems

1. Using the horn theory of the text, deduce the expression for the pressure amplification of the horn whose cross section obeys the law $S = S_0 e^{-ax^2}$.

2. A conical horn with large end open is to be used as a branch in an acoustical conduit. Considering the impedance of the open end to be that of an orifice, calculate the impedance of the throat.

3. Using the result of the preceding problem, calculate the power transmission ratio through the conduit as affected by the conical horn used as a branch.

4. Discuss the theory of an infinite conical horn.

CHAPTER VII

The Filtration of Sound

7·1. General Considerations.—Simple Methods of Filtration.— In the previous sections of the text we have met illustrations of the filtration of sound. Thus it was found that the Quincke tube (Sec. 3·10) acts to eliminate the transmission of sounds of definite frequency, namely, those for which the difference in length of the two parallel branches is an odd multiple of $\lambda/2$ and those for which the sum of the two branch lengths is equal to any integral multiple of λ (subject to a certain subsidiary condition which may be referred to in the section cited). The tube is highly selective in contrast to a filter that removes a relatively large band of frequencies.

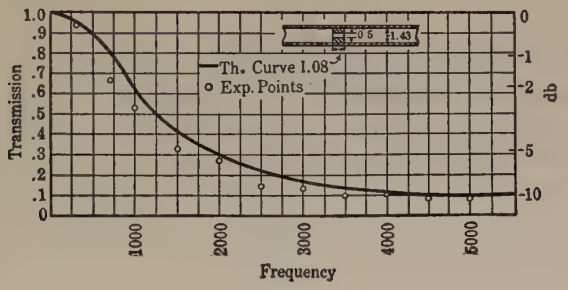

Fig. 7·1.

In Section 3·6 it was found that the insertion of a channel in an acoustic conduit, whether constricting or expanding in nature, renders the conduit selective. It suffices to recall the general expression for the power transmission P_r (eq. (3·61)) to note the dependence on the frequency. Experiments have been carried out to check this effect. The following figure (Fig. 7·1) shows the comparison between the theoretical and experimental results in the case of a constriction 1.08 cm long and 0.5 cm in diameter in a conduit 1.43 cm in diameter. The progressive falling off in the transmission as the frequency increases is clearly evident. The difference between the curve and the experimental points is probably due to the omission of viscosity in the derivation of the formula.

As a matter of fact, inspection of eq. (3·61) shows that if the range of frequencies were increased the transmission curve would display a series of peaks alternating with hollows, the former corresponding to the frequencies for which $\sin kl = 0$. The diagram presented here thus shows only the low frequency part of the actual behavior of the tube. It will be noted that the transmission is at no place reduced actually to zero. In Fig. 7·2 are presented the results of

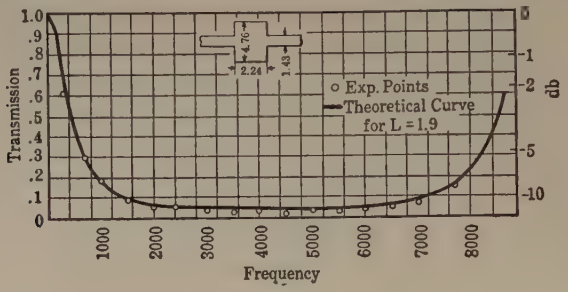

Fig. 7·2.

an experiment with an expansion channel 2.24 cm long and 4.76 cm in diameter in a conduit of the same diameter as that mentioned immediately above. There will be noted a failure to transmit more than 5% over a frequency range from 2000–6000 cycles. But again the transmission is not reduced even approximately to zero and the variation of intensity near the selected frequencies is not rapid. In this case, also, extension to higher frequencies would show an alternate series of peaks and hollows.

In Chapter V we also noticed the selective effect of the Helmholtz resonator used as a branch (see Fig. 5·3). But the selectivity is not very marked except over a narrow band of frequencies. Experiments on an orifice as a branch give results of a similar nature, in particular showing reduced transmission at low frequencies. But again the filtration is only partial. On the basis of these results it might be supposed that placing several selective devices in series (i.e., successive orifices, etc.) would introduce greater selectivity. An experiment was performed to illustrate this point. In a conduit 0.556 cm in diameter, branch orifices of diameter 0.15 cm were placed 1.5 cm apart. The transmission curves are given in Fig. 7·3. For a few orifices in line there was no startling improvement, as is indicated in curves *A*, *B* and *C*. But an increase in the number of

orifices to *eight* brought about a fairly complete cut-off of the transmission below 2600 cycles while for higher frequencies the transmission rose to 80% or more. It seems clear that a really new phenomenon is involved here, needing more detailed study in a

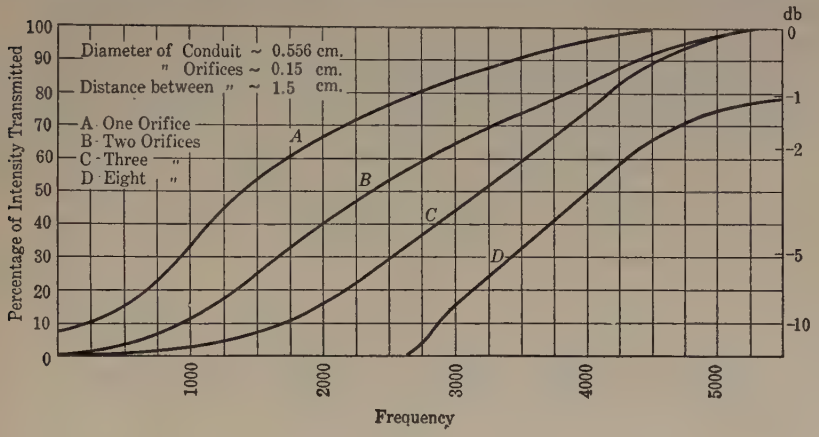

FIG. 7·3.

somewhat different manner from that hitherto employed in discussing transmission problems. Historically, this study developed in the manner related in the following section, having been suggested by the theory of the electrical wave filter, and not by any acoustic experiments.

7·2. Theory of the Acoustic Filter.—In this section there will be developed a formal theory of acoustic filtration.[1] We shall begin with certain definitions and assumptions. An acoustic line is defined as a bounded region forming a tube or channel and capable of transmitting sound waves in the direction of the tube or channel only. The accompanying diagram (Fig. 7·4) is a schematic representation of an acoustic filter in the form that will be studied here. The acoustic line is AG, here taken as a portion of an infinite line. Herein are inserted in series the equal impedances Z_1. At the points A, C, E, G, there are branch lines AB, CD, etc., containing the equal impedances Z_2. For the present we shall define as a

[1] This is the lumped impedance theory first presented by G. W. Stewart, Phys. Rev., **20**, 528, 1922. More recently W. P. Mason (Bell Tech. Jl., **6**, 258, 1927) has developed a more general theory, the equivalent of which will be presented in Appendix IV. The results of both theories will be freely used in the following sections.

section of the filter the portion between A and C, or C and E, etc. In order that a negligible change of phase may take place in each section so defined, we shall make the assumption that the length of each is small compared with the wave length, the same applying to each branch length, also. Later a more refined definition of section will be given. The branches will be assumed to terminate in a volume of gas otherwise at rest, so that the pressure at all the termini is approximately the same and in value zero excess pressure.

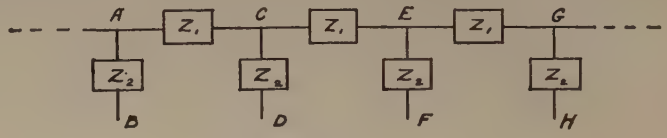

FIG. 7·4

These are the only general conditions imposed. The impedances Z_1 and Z_2 are thus far arbitrary, and we have to investigate the specific conditions they must satisfy in order that the line may possess filtering properties.

We make the following further assumptions:

(1) The volume current at any part of the line will always be expressed by $\dot{X}e^{i\omega t}$, where $\dot{X}$ is complex.

(2) The acoustic impedance will be defined as usual by $Z = p/\dot{X}e^{i\omega t}$.

(3) The algebraic sum of the volume currents at any junction point A, C, E, etc., is zero.

(4) The excess pressure at each junction point is the same for both main line and branch (i.e., continuity is assumed).

(5) The direction of positive current is from left to right.

Let us fix our attention on the section from C to E. The volume current into C from the left is $\dot{X}_{n-1}$. That out of C to the right is $\dot{X}_n$. Consequently by the continuity assumption (3) above, the current out of C to D is $\dot{X}_{n-1} - \dot{X}_n$. And hence the excess pressure at C in terms of Z_2 is $Z_2(\dot{X}_{n-1} - \dot{X}_n)$. Similarly that at E is $Z_2(\dot{X}_n - \dot{X}_{n+1})$. But the difference between the excess pressures at C and E expressed in terms of Z_1 is $Z_1\dot{X}_n$. We therefore have the equation

$$Z_2(\dot{X}_{n-1} - \dot{X}_n) = Z_2(\dot{X}_n - \dot{X}_{n+1}) + Z_1\dot{X}_n, \qquad (7\cdot1)$$

whence there follows

$$\dot{X}_{n+1} - \left(2 + \frac{Z_1}{Z_2}\right)\dot{X}_n + \dot{X}_{n-1} = 0. \qquad (7\cdot2)$$

We now assume for convenience that the filter is an infinite one and denote the impedance of the infinite network to the right of any section by Z_∞. This implies that all Z_∞ are identical. We then have in terms of the main line impedance for the excess pressures at C and E

$$\begin{aligned} p_C &= \dot{X}_n(Z_1 + Z_\infty), \\ p_E &= \dot{X}_{n+1}(Z_1 + Z_\infty). \end{aligned} \tag{7·3}$$

It should be recalled that we are assuming that the excess pressure is zero at the termini B, D, F, etc. It now follows from (7·3) that

$$\frac{p_C}{\dot{X}_n} = \frac{p_E}{\dot{X}_{n+1}}. \tag{7·4}$$

Now if we substitute into the above

$$p_C = Z_2(\dot{X}_{n-1} - \dot{X}_n)$$

and

$$p_E = Z_2(\dot{X}_n - \dot{X}_{n+1}),$$

we have at once the simple but important relation

$$\frac{\dot{X}_{n+1}}{\dot{X}_n} = \frac{\dot{X}_n}{\dot{X}_{n-1}}. \tag{7·5}$$

This constant complex ratio will be called e^Y, where Y is complex. Substitution into the eq. (7·2) then yields

$$e^Y + e^{-Y} = 2 + Z_1/Z_2$$

or

$$\cosh Y = 1 + \tfrac{1}{2}Z_1/Z_2. \tag{7·6}$$

Now if Y is a pure imaginary, e^Y is a pure circular function and $\dot{X}_{n+1}$ and $\dot{X}_n$, etc., differ only in phase. That is, there is no progressive attenuation in this case. But if Y is a complex number of the form $a + ib$, the volume current as shown by (7·5) will suffer progressive attenuation as the sound passes through the filter line. The condition that Y shall be a pure imaginary is that cosh Y (which is the cosine of $-iY$) shall lie between $+1$ and -1. That is, we have

$$1 > (1 + \tfrac{1}{2}Z_1/Z_2) > -1. \tag{7·7}$$

It therefore follows that the region in which we get *no* attenuation is

bounded by the limits

$$Z_1/Z_2 = 0$$

and

$$Z_1/Z_2 = -4. \tag{7·8}$$

Outside of this region there *will* be attenuation. This then opens up the possibility of constructing an acoustic filter to specification. The question that arises is, what should be used in practice for Z_1 and Z_2? We shall discuss first some ideal cases.

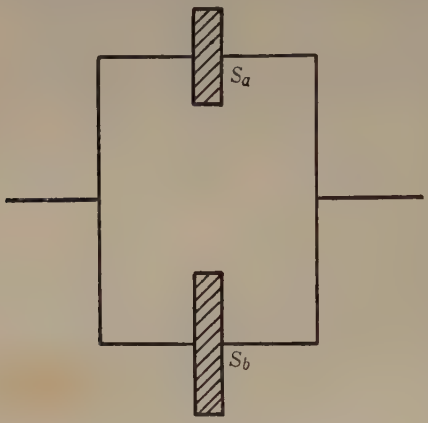

Fig. 7·5.

Let us find the resultant impedance of an inertance and a capacitance placed in parallel. Consider two idealized diaphragms a and b, of which the first has a mass m_a and negligible stiffness, while the second has stiffness f_b and negligible mass. In both cases the damping is neglected. Let the effective cross sectional areas be S_a and S_b. The two diaphragms are arranged in parallel, so that they are operated by the same excess pressure, viz., $Pe^{i\omega t}$. The equations of motion of the diaphragms are then respectively

$$\left.\begin{aligned} m_a\ddot{\xi}_a &= S_a P e^{i\omega t}, \\ f_b\xi_b &= S_b P e^{i\omega t}, \end{aligned}\right\} \tag{7·9}$$

where the displacements have been denoted by ξ_a and ξ_b. Using volume currents $X_a = S_a\xi_a$, etc., instead, we have

$$\left.\begin{aligned} \frac{m_a}{S_a^2}\ddot{X}_a &= P e^{i\omega t}, \\ \frac{f_b}{S_b^2}X_b &= P e^{i\omega t}. \end{aligned}\right\} \tag{7·10}$$

The volume current in the branch containing the inelastic diaphragm is then obtained by simple integration (making the integration constant zero) and is given by

$$\frac{m_a}{S_a^2}\dot{X}_a = \frac{P}{i\omega}e^{i\omega t}. \tag{7·11}$$

Similarly the volume current $\dot{X}_b$ is given by

$$\frac{f_b}{S_b^2}\dot{X}_b = i\omega P e^{i\omega t}. \tag{7·12}$$

Now at the junction of the two branches we have continuity and hence

$$\dot{X}_a + \dot{X}_b - \dot{X} = 0, \tag{7·13}$$

where $\dot{X}$ is the current in the line (with the positive direction measured away from the junction). On substitution we have

$$\left(\frac{1 - MC\omega^2}{i\omega M}\right) P e^{i\omega t} = \dot{X}, \tag{7·14}$$

wherein we have set $M = m_a/S_a^2$ and $1/C = f_b/S_b^2$. M is the inertance of the inelastic system and C is the capacitance of the elastic system, as we recall from the definitions of Section 2·3. For the impedance of the two in parallel we have

$$Z = \frac{P e^{i\omega t}}{\dot{X}} = \frac{i\omega M}{1 - MC\omega^2}. \tag{7·15}$$

Let us now suppose that the impedances Z_1 and Z_2 of the filter line are *each* composed of such a parallel arrangement of inertance and capacitance. Then of the conditions (7·8) the first can be satisfied only by $Z_2 = \infty$ (if we disregard very high frequencies and do not allow M to become negligible). This leads (by (7·15)) to a frequency

$$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{1}{M_2 C_2}}, \tag{7·16}$$

as one limiting frequency of the non-attenuation region. M_2 and C_2 are the inertance and capacitance, respectively, making up the impedance Z_2. The second condition ($Z_1/Z_2 = -4$) leads to a frequency

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{M_1 + 4M_2}{M_1 M_2(4C_1 + C_2)}}, \tag{7·17}$$

where M_1 and C_1 belong to Z_1. Thus between ν_1 and ν_2 there lies a non-attenuation region which may by appropriate choice of Z_1 and Z_2 be made of observable magnitude and hence of utility.

We next consider the placing of inertance and capacitance in series. In every part of the line the volume current is the same,

i.e., $\dot{X}_a = \dot{X}_b$, if we continue to use the analogy introduced above. Denoting the excess pressure over the inertance by P_a and that over the capacitance by P_b, we have (recalling (7·11) and (7·12))

$$\frac{P_a S_a^2}{i\omega m_a} = \frac{i\omega P_b S_b^2}{f_b}, \tag{7·18}$$

which becomes on substitution of M and C

$$\frac{P_a}{i\omega M} = i\omega C P_b = \dot{X}$$

or

$$\frac{P_a + P_b}{i\omega M + \frac{1}{i\omega C}} = \dot{X}, \tag{7·19}$$

whence the impedance of the two in series will be

$$Z = \frac{P_a + P_b}{\dot{X}} = i\left(M\omega - \frac{1}{\omega C}\right). \tag{7·20}$$

Suppose that Z_1 and Z_2 are both made in this way, i.e.,

$$Z_1 = i\left(M_1\omega - \frac{1}{\omega C_1}\right), \qquad Z_2 = i\left(M_2\omega - \frac{1}{\omega C_2}\right).$$

The frequency limits given by (7·8) are then easily calculated to be

$$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{1}{M_1 C_1}} \tag{7·21}$$

and

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{4C_1 + C_2}{C_1 C_2(M_1 + 4M_2)}}. \tag{7·22}$$

The Helmholtz resonator provides a good example of inertance and capacitance in series (provided dissipation be neglected or radiation not allowed to take place). From our previous work (Sec. 2·3) we recall that the inertance of the resonator (which is due to the mass of vibrating air in the opening) is

$$M = \rho_0/c_0,$$

while the capacitance, which arises from the stiffness of the air chamber, is

$$C = V/\rho_0 c^2,$$

where V is the volume of the chamber and c_0 is the conductivity of the opening. These values when substituted into (7·21) and (7·22) give the limits of the non-attenuation region, when both Z_1 and Z_2 consist of Helmholtz resonators (an impractical arrangement, of course, since a Helmholtz resonator is a closed vessel and cannot be put in series with anything).

Since it is essential to have the transmission occur through Z_1, it is evident that the simplest construction for Z_1 would be a tube. Now what are the inertance and capacitance of a tube? The former seems simple enough, being equal to the mass of the air in the tube (or section thereof) divided by the square of the cross-sectional area (Sec. 2·3). In Section 2·6 we saw that a tube has capacitance which, indeed, is of magnitude $S/\rho_0 c^2$ per unit length, where S is the area of cross section. But this capacitance is not localized but distributed. Hence there arises the difficulty that we are not able to separate the impedance elements for a tube.

It is not necessary here to go in detail into the reasoning for and the experimental verification of the final selection of the approximate values for Z_1. It will be sufficient to remark that if the length of the tube comprising Z_1 is short compared with the wave length, the fluid in the tube may be said to move as a whole without phase difference, whence the inertance will be clearly $\rho_0 l/S^2$, where l is the length of a section. The actual construction of filters based on this theory will be discussed in the following three sections.[1]

It should again be emphasized that the foregoing theory is strictly limited by the restrictions imposed, namely, that the air in each section moves as a whole, since the length of each one is short compared with the wave length. It is obvious that independently of the choice of Z_1 and Z_2, we can not expect the theory to work for high frequencies. And of course we have neglected viscosity dissipation.

A more accurate theory of filtration would not lump the impedances as is done in the foregoing, but would consider the wave transmission through the main line as affected by the presence of the branches, in the way in which the problems of Chapter V have been solved. This procedure has actually been carried through by Mason (loc. cit., above). A detailed analysis of this kind will be

[1] For further details see G. W. Stewart, Phys. Rev., **20**, 528, 1922.

found in Appendix IV. The principal results only will be stated here.

If we denote the volume current at the beginning of the mth section of an infinite filter by $\dot{X}_m$, the attenuation relation on this theory takes the form

$$\dot{X}_{m+1}/\dot{X}_m = e^{-iW}, \tag{7·22a}$$

where

$$\cos W = \cos kl_1 + \frac{iZ}{2Z_2} \sin kl_1. \tag{7·22b}$$

In the formula (7·22b) l_1 = length of one section, while Z_2 is the branch impedance at the junction, as before. Z is *not* the same as Z_1. Rather we have here $Z = \rho_0 c/S$, the acoustic resistance of the plane wave in the conduit. The transmission region is then that for which W is real, i.e., that for which we have

$$-1 < \cos kl_1 + \frac{iZ}{2Z_2} \sin kl_1 < +1. \tag{7·22c}$$

If the excess pressure at the beginning of the mth section is p_m, we have

$$p_{m-1}/\dot{X}_{m-1} = p_m/\dot{X}_m = \text{const.} = Z_0,$$

where Z_0 may be defined as the *characteristic* impedance of the filter. Analysis shows that

$$Z_0 = Z\sqrt{\frac{1 + iZ/2Z_2 \cdot \tan (kl_1/2)}{1 - iZ/2Z_2 \cdot \cot (kl_1/2)}}. \tag{7·22d}$$

Comparison between the two filter theories will be made in connection with the discussion of the construction of definite types of filters. It will be found that in general the more exact theory is of no greater advantage in practice than the lumped impedance theory. In fact the application to the actual construction of filters is more easily made on the basis of the latter. For this reason and also because of its greater physical suggestiveness and mathematical simplicity the main emphasis in this chapter has been placed on the lumped impedance point of view.

7·3. The Construction of Acoustic Filters.—I. Low Frequency Pass.—The design of a possible low frequency pass filter is shown in the following schematic diagram (Fig. 7·6). The main line or Z_1 sections are formed by the tube through which the transmission

takes place. The branch or Z_2 sections are Helmholtz resonators, so that Z_2 is formed of an inertance and capacitance in series. In

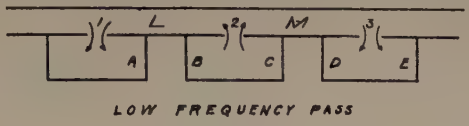

FIG. 7·6.

this case Z_1 will be assumed to be made up of inertance only, i.e., $C_1 = 0$. Thus (referring to (7·15) and (7·20)) we have

$$\left.\begin{aligned} Z_1 &= i\omega M_1, \\ Z_2 &= i\left[\omega M_2 - \frac{1}{\omega C_2}\right], \end{aligned}\right\} \tag{7·23}$$

whence

$$Z_1/Z_2 = \frac{\omega^2 M_1 C_2}{\omega^2 M_2 C_2 - 1}. \tag{7·24}$$

Since M_2 can not be infinite nor M_1 zero, the condition $Z_1/Z_2 = 0$ can be satisfied only by $\omega = 0$; hence, one of the limiting frequencies of the non-attenuation region is $\nu_1 = 0$. The other one comes out to be

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{4}{C_2(M_1 + 4M_2)}}.$$

The inertance of each line section is given by

$$\begin{aligned} M_1 &= \frac{\text{mass of air in tube per section}}{S_1^2} \\ &= \frac{\rho_0 l_1}{S_1}, \end{aligned} \tag{7·25}$$

where l_1 is the length of one section. The inertance of the branch is

$$M_2 = \rho_0/c_0, \tag{7·26}$$

where for c_0, the conductivity of the orifice, we write (see Sec. 2·4, eq. (2·33))

$$c_0 = \frac{\pi a^2}{L + \frac{\pi a}{2}}, \tag{7·27}$$

a being the radius of the orifice and L the channel length. Finally

$$C_2 = V_2/\rho_0 c^2, \tag{7·28}$$

in which V_2 is the volume of the resonator chamber. Making the substitution, we have for the range of no attenuation

$$\nu_1 = 0 \qquad \text{to} \qquad \nu_2 = \frac{c}{\pi}\sqrt{\frac{S_1}{l_1 V_2}\left(\frac{1}{1+\frac{4S_1}{l_1 c_0}}\right)}, \qquad (7\cdot29)$$

wherein S_1 = area of cross section of the main line.

An illustration of this type of filter is presented in Fig. 7·7. This particular filter has the following constants: $l_1 = 1.67$ cm, r_1 = radius of the main line = 0.75 cm, r_2 = radius of surrounding chamber = 1.30 cm, $a = 0.126$ cm, $V_2 = 4.36$ cm^3. The con-

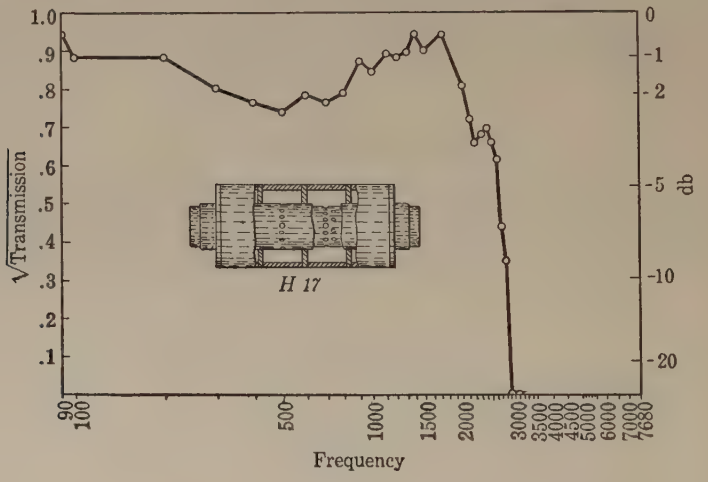

Fig. 7·7.

ductivity c_0 computed by eq. (7·27) turns out to be .141 for each aperture and, since there are 16 apertures in this particular model, the total c_0 to be substituted into eq. (7·29) is (.141)16 = 2.26. We note in passing that conductivities in parallel are additive, as in the analogous electric case. The cut-off frequency calculated from eq. (7·29) is found to be $\nu_2 = 3175$ cycles. Examination of the experimental curve shows good agreement between the theory and the actual cut-off. After the cut-off there is no audible frequency until 5200 cycles is reached, and even here the transmission is negligible. The whole range of inaudibility is found to be 66% of the cut-off frequency, and the average percentage transmission in the non-attenuation region is 65%. This transmission seems to depend for a

given number of sections on the ratio between the outside chamber radius and the main line radius (both measured from the axis of the line). Observations on a large number of filters [1] show that the transmission varies inversely as this ratio. On the other hand, the range of inaudibility above the cut-off increases with this ratio. It must be pointed out, however, that the observations referred to concern the transmission when the conduits leading to and from the filter have the same diameter as the conduit of the filters. As will be shown in Section 7·8, this is not the optimum condition for transmission at a given frequency. Other things being equal, increasing the number of apertures into the side chamber increases the range of inaudibility. As one would expect, it also increases the cut-off frequency. Although there are a number of factors involved, it is possible to make a successful design for a low frequency pass filter.

The following figure (Fig. 7·8) shows in greater detail than the previous one the construction of a low frequency pass filter of different size and characteristics, and may prove of help in interpreting the dimensions given for the model just discussed.

According to the theory of Mason (Appendix IV) the transmission region is given by the condition (see eq. (7·22*c*))

$$-1 < \cos kl_1 + \frac{iZ}{2Z_2} \sin kl_1 < +1. \qquad (7\cdot29a)$$

The frequency limits are then given by the transcendental equations

$$\frac{iZ}{2Z_2} = -\cot (kl_1/2), \qquad \frac{iZ}{2Z_2} = \tan (kl_1/2), \qquad (7\cdot29b)$$

analogous to eqs. (7·8) on the lumped impedance theory. If kl_1 is so small that we can neglect higher powers than the first (i.e., short sections and low frequencies), the eqs. (7·29*b*) become identical with (7·8), provided we set $Z_1 = ikl_1\rho_0 c/S = i\omega M_1$, where M_1 is as above. But this is just what we chose for Z_1 in this section. Hence we see that for the low frequency pass type of filter the two points of view give to a close approximation the same result.

7·4. The Construction of Acoustic Filters.—II. High Frequency Pass.—The accompanying schematic diagram (Fig. 7·9) indicates

[1] See G. W. Stewart, Phys. Rev., **20**, 546, 1922, for details on filter construction.

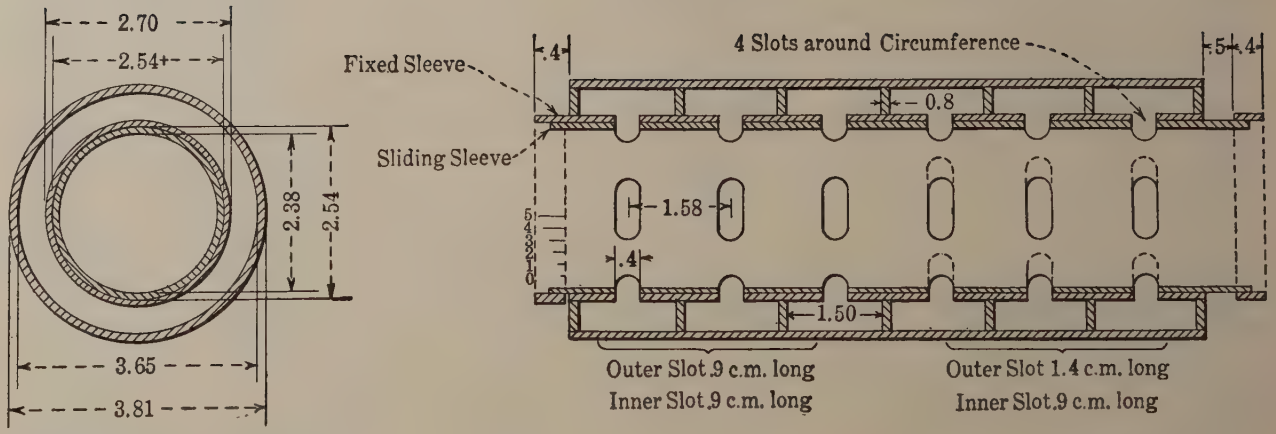
2.70
2.54
2.38
2.54
3.65
3.81
All Dimensions in Centimeters
Fixed Sleeve
Sliding Sleeve
.4
4 Slots around Circumference
.5
.4
0.8
1.58
.4
1.50
Outer Slot .9 c.m. long
Inner Slot .9 c.m. long
Outer Slot 1.4 c.m. long
Inner Slot .9 c.m. long

FIG. 7·8.

the design of a high frequency pass filter. The branches are here simple orifices, possessing inertance only. That is, $Z_2 = i\omega M_2$. The question arises, what shall one take for Z_1? The previous section assumed that Z_1 was a simple inertance. But this assumption here will lead to nothing. It seems necessary to introduce capacitance in the main conduit. There is indeed some justification

FIG. 7·9.

for this step. The orifices which compose the branch lines are very short and consequently the pressure gradient in them is much greater than that in the line. Hence we may take it that the particle velocity in M_2 is much greater than that in the line. Effectively, it is much as if the particles in the line were at rest relative to those in the orifices. But then the line would constitute a capacitance for the orifices. The question is, how to introduce it? If we consider the line to be made up of inertance and capacitance in parallel, we then have

$$Z_1 = i\left(\frac{M_1\omega}{1 - M_1C_1\omega^2}\right),$$

and consequently

$$Z_1/Z_2 = M_1/M_2(1 - M_1C_1\omega^2). \qquad (7{\cdot}30)$$

Introducing $M_1 = \rho_0 l_1/S$, $M_2 = \rho_0/c_0$, and $C_1 = V_1/\rho_0 c^2$, where V_1 = volume of one section of the line, eq. (7·30) becomes

$$Z_1/Z_2 = c_0/kS \cdot kl_1/(1 - k^2l_1^2).$$

This will be zero only for k (i.e., ν) equal to ∞. The other frequency limit will be given by

$$c_0/2kS = -2(1 - k^2l_1^2)/kl_1. \qquad (7{\cdot}31)$$

But from (7·29*b*) we see that the more accurate theory gives for this same limit

$$c_0/2kS = \tan(kl_1/2), \qquad (7{\cdot}32)$$

and hence the present theory will give correct results only if

$$\tan(kl_1/2) = -2(1 - k^2l_1^2)/kl_1, \qquad (7{\cdot}33)$$

which is not an identity but approximately satisfied if $kl_1/2$ is in the immediate neighborhood of $\sqrt{\frac{1}{3}}$. It is evident [1] that the assumption of (7·30) will not do in general. However, let us suppose that capacitance only is attributed to the conduit, a not improbable assumption. This would lead to $Z_1 = -i/C_1\omega$. But following the suggestion of the more accurate theory, we let $Z_1 = -4i/C_1\omega$, i.e., assign only one quarter the total capacitance per section as given above. The upper frequency limit is then still $\nu = \infty$, while the lower limit is given by the relation

$$c_0/2kS = kl_1/2, \tag{7·34}$$

which agrees to within 10% with (7·32) if $kl_1/2 < 0.50$. The percentage error in the frequency obtained by using (7·34) will be about half this. The lower limit is then

$$\nu_2 = \frac{c}{2\pi}\sqrt{\frac{c_0}{V_1}}. \tag{7·35}$$

The frequencies on the lumped impedance theory are then $\nu_1 = \infty$ and ν_2 as in eq. (7·35). The latter can be used with reasonable success even if still semi-empirical and suggested by the more accurate theory.

It may be well to note here that, while we have assumed consistently that Z_2 is a pure reactance, there may arise cases in which this is not true. For example, if the branch, instead of being a simple orifice, were a long cylindrical tube, Z_2 would have both real and imaginary components. (Recall eq. (5·30), Sec. 5·4.) Such cases are easily treated on the more general theory (Mason, loc. cit., p. 264).

An illustration of the high frequency pass type of filter is presented in the following figure, together with the transmission curve. This particular filter is composed of six sections. The other constants are $l_1 = 10.0$ cm, $r_1 = 0.485$ cm, $a = .139$ cm, $L =$ length of orifice channel $= 0.5$ cm. The conductivity is computed from (7·27) to be $c_0 = 0.0845$. Substitution into (7·31) yields for the cut-off frequency $\nu_2 = 620$ cycles, in good agreement with the observed results. This filter and others of like construction with different

[1] The success with which G. W. Stewart (loc. cit., p. 547) used (7·30) in treating high pass filters is due just to the fact that, in the examples he gives, $kl_1/2$ approximates the above value. See R. B. Lindsay, Phys. Rev., **34**, 652, 1929.

dimensions give good transmission for all frequencies (so far tested) above the cut-off. In the case here cited the average transmission from the cut-off to 7000 cycles is about 80%. The results for a great number of high frequency pass filters studied by one of the

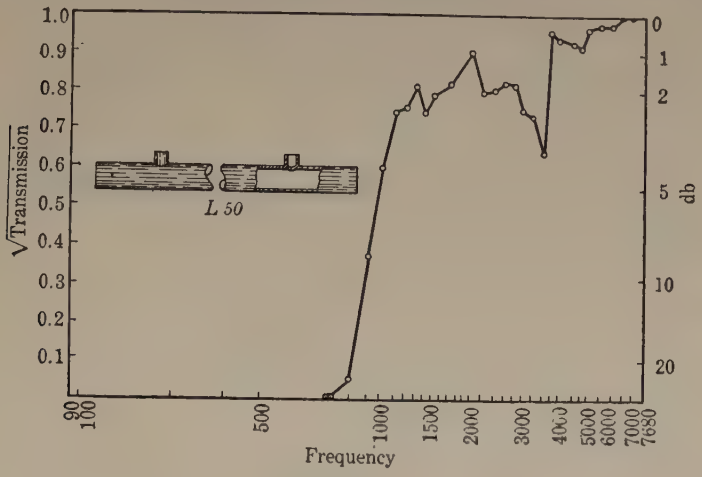

FIG. 7·10.

authors[1] indicate that the actual number of sections is not vital. It should be noted, however, that decreasing the radius of the orifices or increasing their channel length reduces the transmission in the non-attenuation region. This is due to viscosity.

7·5. The Construction of Acoustic Filters.—III. The Single Band Type.—By combining the two types mentioned in the preceding sections, it is possible to construct filters which allow (to a first approximation) the transmission of but a definite band of frequencies. A schematic diagram of such a filter is presented in the following figure (Fig. 7·11). Theoretically Z_1 is an inertance only, while Z_2 is made up of C_2 and M_2 in parallel. But in the practical construction it is necessary to have an orifice leading into the branch. This will then have an inertance which we denote by

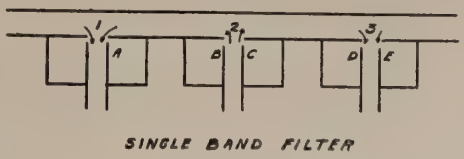

FIG. 7·11.

[1] G. W. Stewart, loc. cit.

M_2'. This will be in series with C_2, with a resultant which from eq. (7·23) we can write as

$$Z_2' = i\left(M_2'\omega - \frac{1}{C_2\omega}\right). \tag{7·36}$$

This must be combined in parallel with

$$Z_2'' = iM_2\omega.$$

Now if the resultant impedance of the side branch is to be denoted as usual by Z_2, we have

$$\frac{1}{Z_2} = \frac{1}{Z_2'} + \frac{1}{Z_2''}, \tag{7·37}$$

whence

$$Z_2 = \frac{i\omega M_2(M_2'C_2\omega^2 - 1)}{M_2C_2\omega^2 + M_2'C_2\omega^2 - 1}, \tag{7·38}$$

while

$$Z_1 = i\omega M_1. \tag{7·39}$$

In the usual way we arrive at the following frequency limits for the non-attenuation region

$$\left.\begin{aligned} \nu_1 &= \frac{1}{2\pi}\sqrt{\frac{1}{C_2(M_2 + M_2')}}, \\ \nu_2 &= \frac{1}{2\pi}\sqrt{\frac{M_1 + 4M_2}{C_2(M_1M_2 + M_1M_2' + 4M_2M_2')}}. \end{aligned}\right\} \tag{7·40}$$

In the evaluation of these expressions we use

$$\left.\begin{aligned} M_1 &= \rho_0 l_1/S_1, \\ M_2 &= \rho_0 l_2/S_2, \\ M_2' &= \rho_0/c_0, \\ C_2 &= V_2/\rho_0 c^2, \end{aligned}\right\} \tag{7·41}$$

wherein S_1 and l_1 are the cross-sectional area and section length of the main line respectively. S_2 and l_2 are the corresponding quantities for the side tube, while the volume V_2 is the volume of the side chamber and c_0 is the conductivity of the orifice into this chamber. On substitution we have

$$\left.\begin{aligned} \nu_1 &= \frac{c}{2\pi}\sqrt{\frac{c_0 S_2}{V_2(l_2 c_0 + S_2)}}, \\ \nu_2 &= \frac{c}{2\pi}\sqrt{\frac{S_2}{l_2 V_2}\left\{\frac{1 + \dfrac{4l_2 S_1}{l_1 S_2}}{1 + \dfrac{S_2}{l_2 c_0} + \dfrac{4S_1}{l_1 c_0}}\right\}}. \end{aligned}\right\} \qquad (7{\cdot}42)$$

As an illustration of this type consider the filter indicated in the diagram. This consists of three sections with $l_1 = 2.66$ cm and

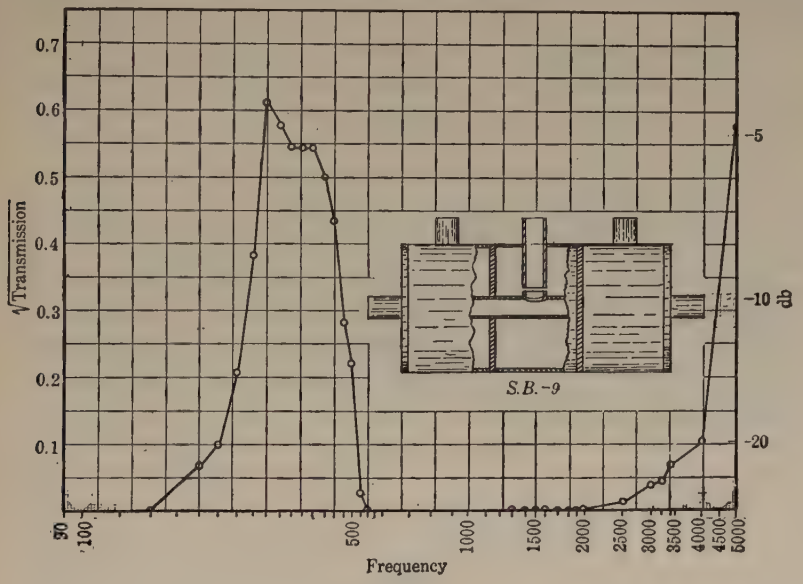

FIG. 7·12.

$r_1 = 0.243$ cm; $l_2 = 2.40$ cm and $V_2 = 22.7$ cm^3. The conductivity $c_0 = 0.455$. The computed values of ν_1 and ν_2 from (7·42) are $\nu_1 = 295$, $\nu_2 = 506$, in good agreement with the measured transmission. The region of inaudibility extends to 1300 cycles, where another (transmission) band begins, the transmission attaining considerable value about 5000 cycles. This additional band, the presence of which is not contemplated in the simple theory above given, will be discussed in a later section. We may note that the inaudibility region is about $3\frac{1}{2}$ times as great as the transmission region. The average transmission is in this case rather small, being only 37%. In all the experiments with single band filters the

average transmission in the first transmission band was found to be comparatively small.

Concerning the bearing of the theory of Mason, already previously discussed (see also Appendix IV), on the single band type filter, it is sufficient to remark that if kl_1 is small the comments at the end of Section 7·3 apply here also and there is little to choose between the two points of view.

7·6. Change of Phase in Filtration.—In the filter theory of Section 7·2 there were developed the expressions (eqs. (7·5) and (7·6))

$$\frac{\dot{X}_{n+1}}{\dot{X}_n} = e^Y, \qquad \cosh Y = 1 + \frac{1}{2}\frac{Z_1}{Z_2}.$$

We may now write e^Y in the general form

$$e^Y = e^{-\alpha}e^{-i\psi}, \tag{7·43}$$

where α and ψ are both real; α is the attenuation constant and ψ the change in phase from section to section. Now reference to the previous sections will indicate that in all cases there discussed the impedances have been pure imaginaries. Hence Z_1/Z_2 has always been real and therefore cosh Y has also been real. Since cosh Y $=$ cosh α cos ψ $+$ i sinh α sin ψ, it follows that we must always have in the cases discussed

$$\sinh \alpha \sin \psi = 0 \tag{7·44}$$

and

$$\cosh Y = \cosh \alpha \cos \psi = 1 + \frac{1}{2}\frac{Z_1}{Z_2}. \tag{7·45}$$

Now in the region of no attenuation we have $\alpha = 0$, whence cosh $\alpha = 1$ and therefore

$$\cos \psi = 1 + \frac{1}{2}\frac{Z_1}{Z_2}. \tag{7·46}$$

Substitution of the values of the ratio Z_1/Z_2 for the limits of the non-attenuation region yields for the first limit cos $\psi = +1$ and for the second limit cos $\psi = -1$. Hence at one limit of the non-attenuation region the change of phase from one section to the next is zero, while at the other limit it is π. Within these limits ψ must vary continuously from the one value to the other, for the functions employed are all continuous. Now in the attenuated region we have $\alpha \neq 0$. Hence from (7·44) it follows that sin $\psi = 0$. Thus ψ

remains constant throughout a region of attenuation and equal to either 0 or π. Moreover from (7·45) it follows that for the region of attenuation

$$\cosh \alpha = -\left(1 + \frac{1}{2}\frac{Z_1}{Z_2}\right), \tag{7·47}$$

where the sign must be chosen to make cosh α positive. The eq. (7·47) can be used to compute α and hence the transmission in the attenuated region, assuming that the transmission is proportional to $e^{-2n\alpha}$, where n is the number of sections. Comparison of theoretical and experimental transmission curves for filters of the kind mentioned in the previous three sections has been carried out by Peacock.[1] The agreement is as good as might be expected, considering the limitations of the theory.

So far as the phase ψ is concerned, we may note that for a low frequency pass filter ψ goes continuously from 0 to π with increasing frequency through the region of no attenuation, while for a high pass filter the variation is from π to 0. Now reference to the values of Z_1/Z_2 for specific filters in the three previous sections indicates that Z_1/Z_2 and hence $1 + \frac{1}{2}Z_1/Z_2$ will pass through infinity with a change in sign somewhere in the attenuation region. This will make $\alpha = \infty$ for this particular frequency and, since cosh α has to remain positive, the theory indicates at this place a discontinuous change in ψ (eq. (7·45)). For the low-frequency pass type for which (see eq. (7·24))

$$1 + \frac{1}{2}Z_1/Z_2 = 1 + \frac{\frac{1}{2}M_1C_2\omega^2}{M_2C_2\omega^2 - 1},$$

the discontinuity will occur for

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{M_2C_2}},$$

and at this point ψ will change discontinuously from π to 0, as is evident from eq. (7·45).

Measurements of ψ have been carried out by one of the authors [2] for the region of no attenuation. A sample curve for a low frequency pass filter is shown in the accompanying figure (Fig. 7·13). The upper full line represents the computed variation in ψ from

[1] See H. B. Peacock, Phys. Rev., **23**, 525, 1924.

[2] G. W. Stewart, Phys. Rev., **23**, 520, 1924.

(7·46) and it is seen that the agreement with the measured values is reasonably satisfactory.

It is of interest to note that ψ/ω denotes the time lag in going from one section to the next. Now since in the non-attenuated region $\cos\psi = 1 + \frac{1}{2}Z_1/Z_2$, it follows that if arc cos $(1 + \frac{1}{2}Z_1/Z_2)$ is approximately a linear function of ω over a certain frequency range, we shall have a filter in which the retardation time is approximately constant over this frequency range. The analogy with electrical retardation lines is obvious.

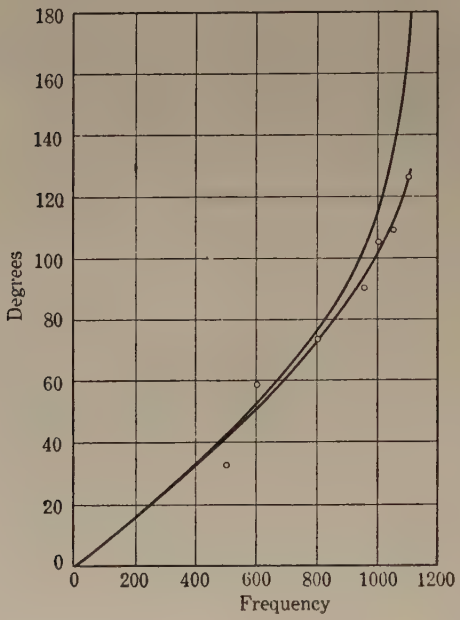

Fig. 7·13.

Essentially the same results as the above are obtained from Mason's theory. Thus recalling (7·22*a*) and (7·22*b*) (see also Appendix IV) and placing $W = a - ib$, where a is the phase factor and b the attenuation coefficient, we have

$$\cos a \cosh b + i \sin a \sinh b = \cos kl_1 + \frac{iZ}{2Z_2}\sin kl_1.$$

Since we have consistently assumed Z_2 to be a pure reactance, cos W is always real. Therefore $\sin a \cdot \sinh b = 0$ and $\cos W = \cos a \cosh b$. In the transmission region there results, since $b = 0$, or $\cosh b = 1$,

$$\cos a = \cos kl_1 + \frac{iZ}{2Z_2}\sin kl_1.$$

In the attenuation region $b \neq 0$ and $\sin a = 0$, so that $a = 0$ or π and

$$\cosh b = \cos kl_1 + \frac{iZ}{2Z_2}\sin kl_1.$$

We still get the discontinuity in the phase a when cosh b becomes infinite.

7·7. **Additional Bands in Acoustic Filters.**—It has already been noted that the attenuation in low frequency pass filters does not extend to indefinitely high frequencies, but that additional non-attenuated bands appear beyond the theoretical cut-off. The theory of these additional bands will not be presented here,[1] but it is of interest to note that their existence can be established and their upper frequency limit fixed in each case by the method outlined in the previous section. For the particular frequency for which the length of one section of a filter of the low frequency pass type is equal to $\lambda/2$, we shall have resonance in the line and hence $Z_1 = 0$. But this means that there will be a region where Z_1 and Z_2 are opposite in sign and $|Z_1| < 4|Z_2|$ again, and so we shall have transmission. This is clearly shown in the following figure (Fig. 7·14). The ability of the theory cited to predict the actual ad-

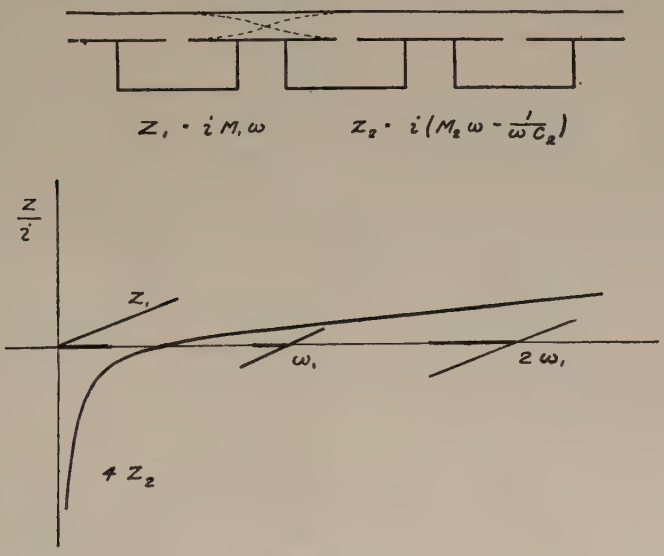

Fig. 7·14.

ditional bands is illustrated in an article[2] by one of the authors.

The general theory of Mason also handles the existence of

[1] See G. W. Stewart, Phys. Rev., **25**, 90, 1925.

[2] G. W. Stewart, Phys. Rev., **25**, 90, 1925.

additional bands very simply. It is only necessary to plot the function

$$\cos W = \cos kl_1 + \frac{iZ}{2Z_2} \sin kl_1,$$

and pick out the regions in which $-1 < \cos W < +1$, which will represent transmission bands. For an example, see Mason, loc. cit., Fig. 3, p. 268.

7·8. Finite Filters.—The theory of Section 7·2 applies to infinite filters, whereas all the filters constructed are, of course, finite. However, all that is necessary in applying the theory to a finite filter is to have the impedance of the conduit at the two ends of a finite number of sections what it would be for an infinite filter. As will later be appreciated, this matching of impedance is always possible only for a selected frequency, and consequently the characteristic action of the infinite filter will not be obtained for all frequencies. Let us now compute this impedance. Consider the single section depicted graphically in the figure. The end impedances are denoted by Z_∞. The idea is this: we want the result of combining the single section with the impedance Z_∞ on the right to give again Z_∞ on the left. Then the result of combining that with the next section will be again Z_∞, etc., so that our procedure in Section 7·2 will then be justified. In order to have this it clearly suffices that

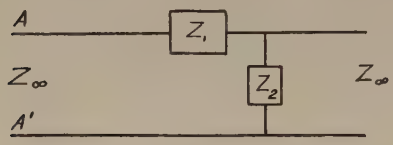

FIG. 7·15.

$$Z_\infty = Z_1 + \frac{1}{\frac{1}{Z_2} + \frac{1}{Z_\infty}},$$

on solving which we arrive at

$$Z_\infty = \tfrac{1}{2}Z_1 \pm \sqrt{Z_1 Z_2 + Z_1^2/4}.$$

As a matter of fact it is somewhat more satisfactory to arrange either the line (i.e., series) impedance or the branch impedance symmetrically. In this way we get the two following diagrams (Fig. 7·16, I and II), in the first of which the line impedance Z_1 is divided into two equal parts in series and in the second of which the branch impedance Z_2 is divided into two equal parts in parallel. In each case the resulting filter network is exactly equivalent to the

previous one presented in Fig. 7·4 of Section 7·2 but has the advantage of greater symmetry. In Fig. 7·16, I, the points $A'B'$, $C'D'$,

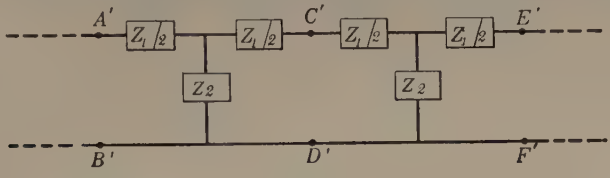

Fig. 7·16 I.

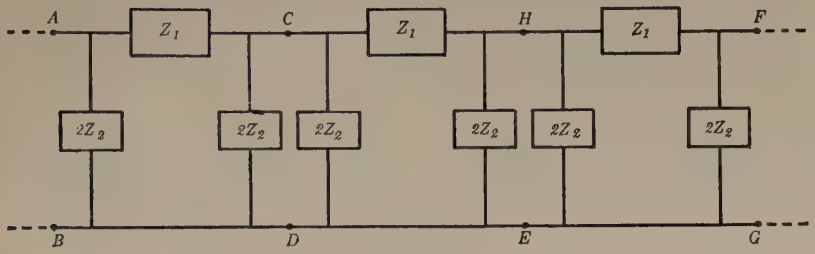

Fig. 7·16 II.

$E'F'$, etc., are termed *mid-series points*, while the points AB, CD, EF, etc., in Fig. 7·16, II, are called *mid-branch points*. We are now in a position to give a more accurate definition of the term *section*. By a *mid-series section* of a filter we shall mean the portion between any two successive mid-series points, i.e., from $A'B'$ to $C'D'$, etc. By a *mid-branch section* of a filter we shall mean the portion between any two successive mid-branch points, i.e., from AB to CD, etc. The impedance for an infinite filter with all sections alike is now the same in either direction from a section, though its value will naturally depend on whether the sections are mid-series or mid-branch. We can compute these values readily. For the mid-series case, consider the section in the following Fig. 7·17. We desire the total impedance to the right of $A'B'$ to be the same as that to the right of $C'D'$, i.e., $Z_\infty^{(s)}$. By the usual scheme of combining impedances we have

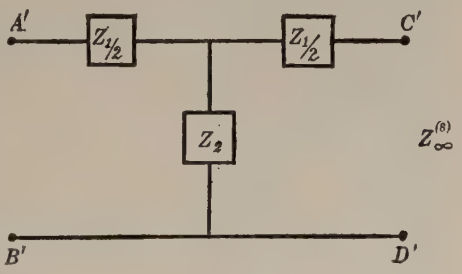

Fig. 7·17.

$$Z_\infty^{(s)} = Z_1/2 + \frac{1}{\frac{1}{Z_2} + \frac{1}{Z_1/2 + Z_\infty^{(s)}}}. \qquad (7{\cdot}48)$$

Solving for $Z_\infty^{(s)}$, we arrive at

$$Z_\infty^{(s)} = \sqrt{Z_1 Z_2 + \tfrac{1}{4} Z_1^2}. \qquad (7{\cdot}49)$$

This will be called the mid-series impedance. The calculation of the mid-branch impedance $Z_\infty^{(b)}$ follows at once from the following Fig. 7·18, in which one mid-branch section is depicted. The usual computation yields here

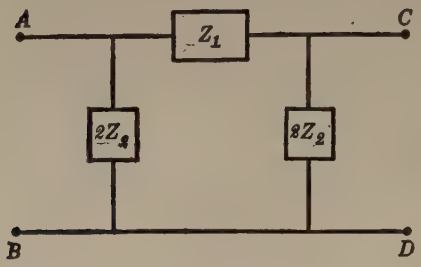

Fig. 7·18.

$$Z_\infty^{(b)} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 + \tfrac{1}{4} Z_1^2}}. \qquad (7{\cdot}50)$$

As a matter of fact the termination of an acoustic filter at a mid-branch point is practically impossible. Hence here eq. (7·50) is of theoretical interest only. We shall concentrate on the mid-series case. It is clear that we have the possibility of constructing a finite filter if the input and output impedances are properly chosen, that is, if the impedance of the acoustic line in which the filter is inserted is properly matched with either the mid-series or mid-branch impedance of the filter, depending on how the insertion is made (i.e., how the filter terminates).

In the theory of Section 7·2 the transmission in the non-attenuation region of the filter was 100%. But the actual experimental tests always show a smaller transmission. It is probable that the reason for most of the difference between the theoretical and actual values is to be found in the improper matching of the impedances of the line and the filter. All we need note here is that the transmission should be materially increased with proper impedance matching. This expectation has been confirmed by experiments in which the filters used terminated at mid-series points. If the acoustic line in which the filter is inserted is a cylindrical tube, we should expect increased transmission for a tube of cross section S, where

$$\frac{\rho_0 c}{S} = \sqrt{Z_1 Z_2 + \frac{1}{4} Z_1^2}, \qquad (7{\cdot}51)$$

since $\rho_0 c/S$ is the point-impedance of an infinite tube of this cross section. Of course, since Z_1 and Z_2 are functions of the frequency, the above relation can hold exactly for but one frequency. Nevertheless, for low-frequency pass filters it can be shown that the radical remains fairly constant over a range of frequencies from zero up to half the cut-off frequency.

The results of an actual test are indicated in the diagram (Fig. 7·19). The "old curve" shows the transmission obtained when the

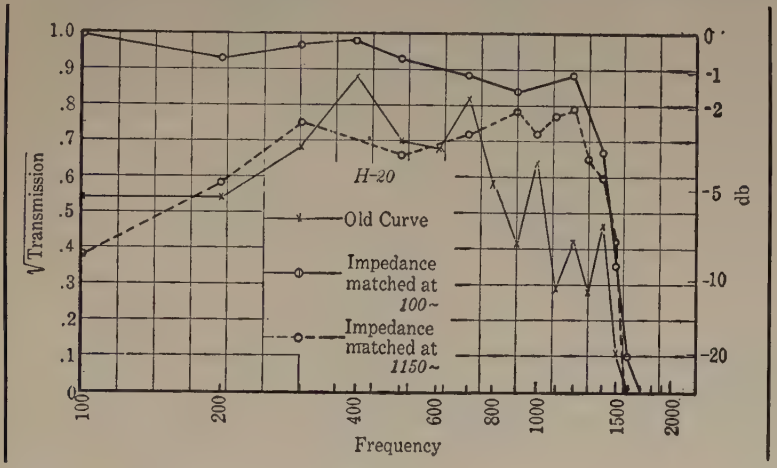

Fig. 7·19.

low-frequency pass filter under consideration was put in a conduit of the *same* cross-sectional area as the filter line itself. The two other curves indicate the result of inserting the filter into conduits with cross-sectional areas calculated from the above eq. (7·51) for the two frequencies 1150 cycles and 100 cycles. The effect on the transmission is very marked and the increase, particularly in the case where the matching was at 100 cycles, rather surprising. It becomes all the more so when it is recalled that in the latter case we have sound traveling from a tube of cross-sectional area 2.43 cm^2 to one of cross-sectional area 0.718 cm^2 with actually relatively increased transmission over a wide range of frequencies. It might be thought that this is inconsistent with the discussion in Section 3·6 of Chapter III on the effect of a constriction on transmission. But it must be remembered that the constriction in the present case is the wave filter, the action of which is quite different from that of a simple tube.

It is necessary to emphasize again that no impedance matching will hold for all frequencies. The best results are to be had for low-pass filters for the reason mentioned above, i.e., the behavior of the radical $\sqrt{Z_1Z_2 + \frac{1}{4}Z_1^2}$ in this case.

In connection with the branch transmission theory it has already been noted (Sec. 7·2, eq. (7.22*d*)) that the so-called *characteristic impedance* has the value

$$Z_0 = Z\sqrt{\frac{1 + iZ/2Z_2 \cdot \tan\,(kl_1/2)}{1 - iZ/2Z_2 \cdot \cot\,(kl_1/2)}}.$$

Now if kl_1 is small, this may be written

$$Z_0 = ikl_1Z\sqrt{\frac{Z_2(1 + iZ/2Z_2 \cdot kl_1/2)}{ikl_1Z}},$$
$$= ikl_1Z\sqrt{\tfrac{1}{4} + Z_2/ikl_1Z}.$$

But from the above the mid-series impedance is

$$Z_\infty^{(s)} = Z_1\sqrt{\tfrac{1}{4} + Z_2/Z_1},$$

and we have already noted in Section 7·3 that for the low-frequency pass and single-band cases to which the above applies $Z_1 = ikl_1Z = ikl_1\rho_0c/S$. Hence to the approximation noted the mid-series impedance of the lumped impedance theory is identical with the characteristic impedance of the transmission theory. Naturally the characteristic impedance is of greater generality. Nevertheless the simpler form of $Z_\infty^{(s)}$ makes it preferable where it is possible to use it. Experimental comparison of Z_0 and $Z_\infty^{(s)}$ with the actually measured impedances will be found in an article by G. W. Stewart and C. W. Sharp.[1] As might have been expected from the above theoretical considerations, the experimental values agree equally well with Z_0 and $Z_\infty^{(s)}$ for small kl_1, while they agree much better with Z_0 for large kl_1 (high frequency pass case).

7·9. Acoustic Wave Filters in Liquids and Solids.—It is to be observed that there is nothing in the preceding theory which would prevent its application to the construction of filters employing liquid media instead of air or other gases. It is conceivable that such liquid filters might be of ultimate advantage in submarine signalling.

The possibility of constructing filters using solid media was early envisaged by one of the authors[2] on the general theory of

[1] Jour. Opt. Soc. of Amer., **19**, 17, 1929.

[2] G. W. Stewart, unpublished.

acoustic wave filters. That is, there is reason to expect that if a metal rod, along which longitudinal waves may be transmitted, is loaded with masses attached at regular intervals, it may show filtering properties. The rod itself thus serves as the acoustic line and the attached masses as the branches. The design is shown in the following schematic diagram (Fig. 7·20). Such arrangements

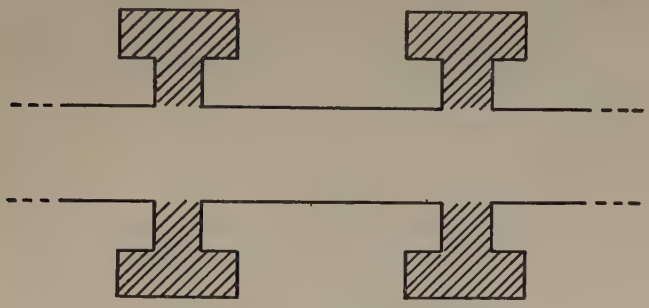

Fig. 7·20.

have been built and their transmission characteristics studied.[1] The results will not be discussed in detail. Nevertheless we should note that the constructions showed reasonably constant transmission over a range of frequencies, followed by a sudden change to high attenuation, quite characteristic of the action of the wave filters previously discussed. The influence of the dimensions of line and branches has been investigated with the following general results:

1. Increase in the volume of the branch decreases the cut-off frequency.
2. Increase in the length of the neck (i.e., the part of the branch where contact is made with the line) decreases the cut-off frequency.
3. Increase in the distance between adjacent branches decreases the cut-off frequency.
4. Additional bands appear at the resonance frequency of the rod between branches.
5. When only one branch is used (i.e., a one-section filter), resonance of the branch takes place with very high attenuation at the corresponding resonance frequency. This reminds one of

[1] See V. C. Hall, Phys. Rev., 23, 116A, 1924; W. D. Crozier, Methods of Testing Acoustic Wave Filters in Solid Media, Master's Thesis, Univ. of Iowa, 1924; H. F. Olson, Action of Acoustic Wave Filters in Solids as Dependent on Dimensions, Master's Thesis, Univ. of Iowa, 1925.

the behavior of the Helmholtz resonator used as a branch in an acoustic conduit, and shows the existence of inertance and capacitance in the branch.

Any reasonably exact theory of this filter would probably be hopelessly complicated. But in a general way the action can be understood by employing the earlier theory as an approximation. Thus if longitudinal waves travel down the rod, there will be impressed on it corresponding radial waves. The displacement of the rod will tend to bring about displacement of the mass in the branch. On account of the large mass of the latter this will cause considerable stress in the portion of the branch where connection is made with the rod. In other words, the connection will display elastic effects and will give rise to capacitance. In addition the mass of the branch will also move as a whole, i.e., it will possess inertance, as will also portions of the rod where the motion takes place as a whole. We may therefore reasonably expect the combination to act as a filter with inertance in the line and inertance and capacitance in series in the branch. Such a filter is a low-frequency pass type, and indeed the results of experiments show that all solid filters constructed on the above design are just of this type. Moreover, it has proved possible to apply the theoretical equations of Section 7·3 to the solid filters studied by getting empirical formulae for the inertance and capacitance, yielding a semi-quantitative understanding of the action of these interesting instruments.

In addition to the filtration of longitudinal waves in solids, it should also be mentioned that the filtration of torsional waves is theoretically just as possible. For this purpose the branch masses may be replaced by sets of vanes with the planes of the vanes containing the axis of the rod along which the torsional waves are transmitted. Experiments by one of the authors [1] show that an arrangement of this kind acts as an efficient filter.

As a matter of fact it should be mentioned here that illustrations of the filtration of sound in solids are to be met with in several connections in applied acoustics. For example, the mechanical lever used in loud speakers performs a filtering action, as does the phonographic needle lever with its diaphragm. Possibilities of mechanical filtering may arise in unexpected quarters. For example, a mechanical filter [2] is at present used for a turntable drive in a disc-recording installation for sound pictures.

[1] G. W. Stewart, unpublished.

[2] Elmer, Bell Laboratories Record, 7, p. 446, 1929.

7·10. Design of Acoustic Wave Filters.—Reference will be made in this section to the design of filters of the types described in Sections 7·3, 7·4 and 7·5 of this chapter. The design of a filter will depend upon the diameter of conduit to be used, the frequency of optimum transmission, the cut-offs, and the ranges of transmission desired. Although the most successful filter will be a matter of trial, since viscosity has been neglected, yet satisfactory filters can readily be constructed by adopting first the conduit, second the separation of orifices, and finally the other specifications determined by computation. The separation of the orifices will determine the position of the additional bands. If an additional band occurs in a region where it is objectionable, it can be removed by an additional filter designed to attenuate in that region. Illustrations of the success of filters in series have been given by one of the authors.[1] It is also possible to make the cut-offs less sharp by the introduction of additional viscosity by means of small orifices. Computations are probably most easily made using the lumped impedance theory as a first approximation (see eqs. (7·8), (7·29), (7·35), and (7·40)).

Questions and Problems

1. A weightless string contains six equally spaced, equally heavy beads. Using the method of Lagrange (see Rayleigh, 2d edition, 1926, Vol. I, p. 172; also Crandall, p. 64), find the natural frequencies of vibration and show that the string acts as a low pass filter.

2. Make a careful comparison of the lumped impedance theory and the transmission theory in so far as the *additional bands* are concerned.

3. Design a high-pass filter having a cut-off at about 1200 cycles and a conduit one-fourth inch inside diameter.

4. If instead of a sharp cut-off high-pass filter you desire a gradual one extending from say 100 to 1200 cycles, what would be your method of procedure to secure such an approximate filter?

5. Design a low-frequency pass filter having a conduit one-fourth inch in inside diameter and a cut-off at about 500 cycles. To what size conduit must this be attached if there is to be no reflection at the junction, assuming, of course, an infinite number of sections?

6. Show graphically where the return bands in a low-pass filter may be expected.

[1] Stewart, Jl. Opt. Soc., **9**, 583, 1924.

CHAPTER VIII

Acoustic Instruments and Measurements

8·1. Measurement of Acoustic Impedance.—Electrical Method.—The first quantitative measurements of acoustic impedance were made by Kennelly and Kurokawa [1] in connection with their study of the acoustic load of an air column on a vibrating diaphragm. As noted in Section 2·5 the definition of acoustic impedance which they use is the ratio of the force to the particle velocity, differing from the one in use throughout the present text. Their method may be summarized as follows: They use a telephone receiver as a generator of sound. The total mechanical impedance of the diaphragm is composed of (1) the actual mechanical impedance, (2) the virtual mechanical impedance due to its motion in the permanent magnetic field, (3) the acoustic impedance of the air behind the diaphragm, (4) the acoustic impedance of the air in front of the diaphragm. It is the last of these which is measured. Thus if the frequency of vibration is maintained constant, while the acoustic load in front is altered (by changing the length of the air column, for example, or substituting a new attachment altogether), the three first impedances will not be altered, while the acoustic impedance in front will change with the load. This in turn will cause a change in the electrical, so-called *motional* impedance of the diaphragm. The latter change can be measured electrically and from it the acoustic impedance. It must be mentioned that it is only in the neighborhood of the resonance frequency of the diaphragm that the acoustic load affects the total mechanical impedance materially, and hence this restricts the application of the method considerably. Any method involving mechanical impedance will obviously suffer from the same handicap. It is therefore hardly suitable to use such a method in the measurement of acoustic impedance over a wide range of frequencies.

8·2. Measurement of Acoustic Impedance.—Acoustical Method.—Another method for the measurement of acoustic impedance has

[1] A. E. Kennelly and K. Kurokawa, Proc. Am. Ac. Arts and Sci., **56**, 1, 1921. See also the extended treatment in Kennelly, Electrical-Vibration Instruments, N. Y., 1923, p. 167.

been devised by one of the authors.[1] This does not involve the measurement of any other mechanical impedance and is strictly acoustical. Moreover the method is absolute in that all measurements can be referred to units of mass, length and time directly, for only the velocity of sound, the density of the medium and certain lengths are involved.

In Section 5·1 we discussed the general theory of a branch line. It was there found that if we denote the excess pressure at the branch junction by p_t (refer back to Fig. 5·1, Sec. 5·1), and that at the same point due to the incident wave from the main conduit by p_i, and the volume current into the branch by $\dot{X}_b$, we can deduce the relation

$$p_t = p_i - \rho_0 c \dot{X}_b / 2S, \tag{8·1}$$

where S is the cross-sectional area of the conduit. The acoustic impedance of the branch is, by definition, $Z = p_t/\dot{X}_b$. Let us have $p_i = p_0 e^{i\omega t}$ and $p_t = p_0' e^{i(\omega t - \epsilon)}$, where ϵ is the phase difference between p_i and p_t. Substitution into eq. (8·1) above then yields

$$p_0/p_0' \cdot e^{i\epsilon} - \rho_0 c/2ZS = 1. \tag{8·2}$$

Now write $Z = Z_1 + iZ_2$ and put the above equation into the form

$$\frac{p_0}{p_0'}(\cos\epsilon + i\sin\epsilon) - \frac{\rho_0 c}{2S} \cdot \frac{Z_1 - iZ_2}{Z_1^2 + Z_2^2} = 1. \tag{8·3}$$

Separating the real and imaginary parts and making the substitutions

$$A = p_0/p_0' \cdot \cos\epsilon - 1,$$
$$B = -p_0/p_0' \cdot \sin\epsilon,$$

we have for the impedance components

$$\begin{aligned} Z_1 &= \rho_0 c/2S \cdot [A/(A^2 + B^2)], \\ Z_2 &= \rho_0 c/2S \cdot [B/(A^2 + B^2)], \end{aligned} \tag{8·4}$$

whence the determination of Z_1 and Z_2 rests on the ability to measure p_0/p_0' and the difference of phase ϵ.

The experimental method is indicated in the following diagram (Fig. 8·1). The main conduit is the tube AD with a telephone

[1] G. W. Stewart, Phys. Rev., **28**, 1038, 1926.

receiver T_1 as the source of sound. At C there is a place for the attachment of the branch whose impedance is to be measured. The bent tube F slides within the tube AD. There is a side tube B also connected at the end to a telephone receiver T_2, and connected to F via a pair of stethoscope binaurals, the attachments being made at G and E. A vacuum tube oscillator is used to excite the source of sound. To secure effective damping of the reflected wave,

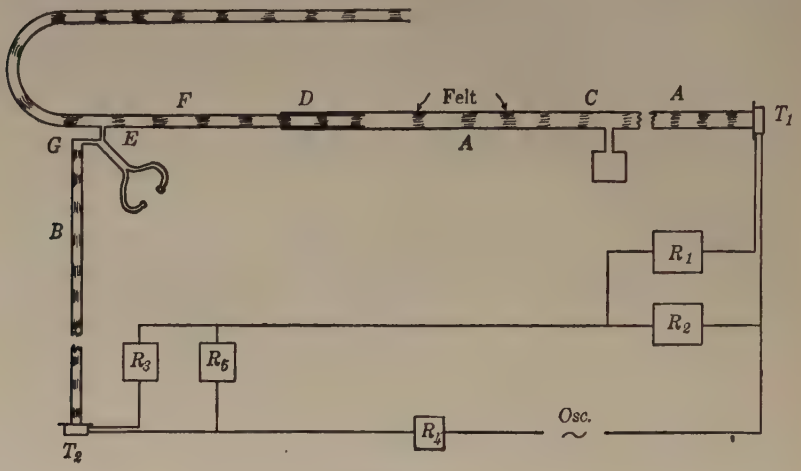

FIG. 8·1.

tufts of hair felt are introduced throughout the tube lengths. The non-inductive resistances R_1 and R_3 are high in comparison with the telephone impedances. R_2 is fixed at a convenient value. The observations are taken as follows: with the branch at C removed and the hole closed the position of the tube F and the value of the resistance R_5 are adjusted (this adjustment being unique) until no sound is heard in the stethoscope, indicating that the amplitudes from E and G are equal and opposite in phase. The branch is then attached at C and the process repeated. The ratio of the pressure amplitudes in F under the two different conditions is given by the ratio of the currents in T_2, and hence by the ratio of the two values of R_5. This will also be the ratio of the two pressure amplitudes at C, i.e., p_0/p_0'. The phase difference ϵ can be computed from the difference between the two settings of F. That is, let d be the distance between the first position and the second, measured positively towards T_1; then $\epsilon = 2\pi d/\lambda$, where λ is the wave length of

the sound used. This method has proved very satisfactory for the middle range of frequencies. To get the same order of accuracy for very high and very low frequencies, a more powerful source of sound is needed. For further details and a discussion of sources of error, reference may be made to the article mentioned at the beginning of the section.

The method has been applied to the measurement of the impedance of attachments of all kinds, including tubes and horns. As illustrations we have already considered the cases of the conical, exponential and hyperbolic horns in Sections 6·3, 6·4, 6·5 (Figs. 6·7, 6·10, and 6·12). We shall note here the results of the measurement for an orifice and a Helmholtz resonator. These are given in Figs. 8·2 and 8·3. In each case the points represent the experimental

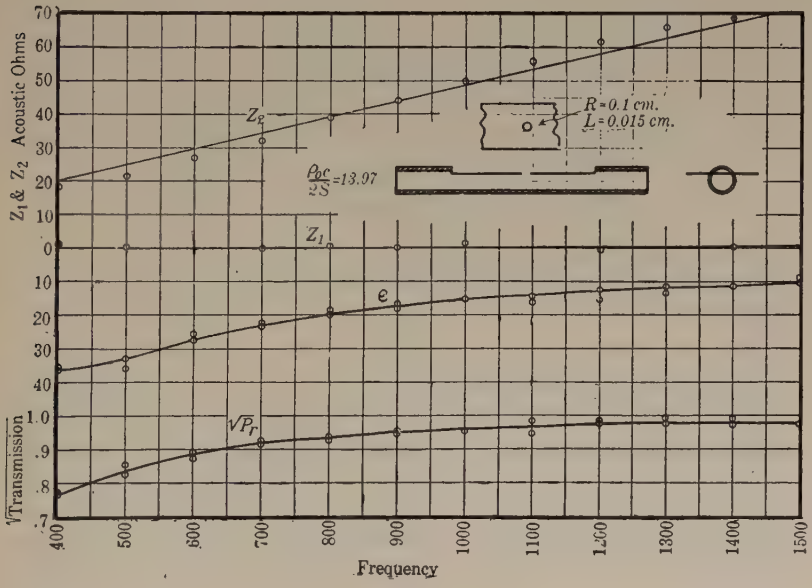

Fig. 8·2.

values of Z_1 and Z_2 as computed from eqs. (8·4) above, while the continuous curves represent the values theoretically computed from the branch line theory of Chapter V (recall eqs. (5·20), (5·21), (5·17) and (5·18)). The units, being c.g.s. throughout, may be defined as "acoustic ohms." The transmission referred to in the figures is that through the main conduit, while the "power" is the

value of $p_0'^2 Z_1/(Z_1^2 + Z_2^2)$.[1] It is the relative power output of the attachment, assuming that the power input in the main conduit is constant. (See Sec. 6·8 for justification.)

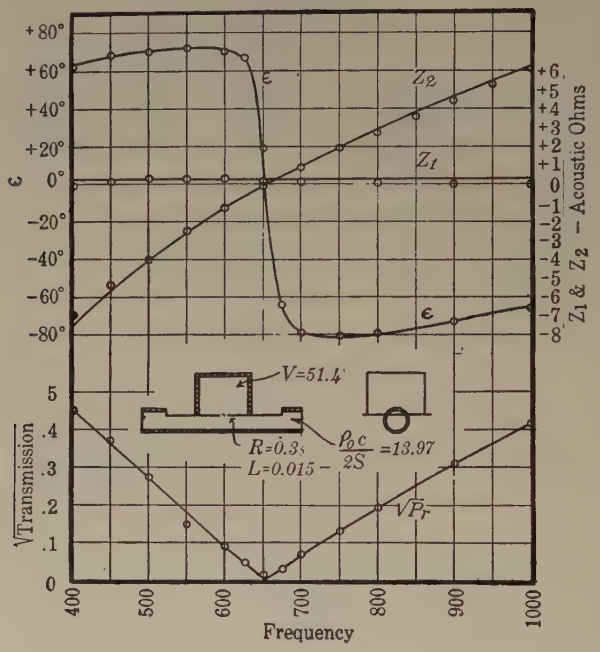

Fig. 8·3.

8·3. Measurement of Acoustic Impedance.—Other Methods.— From the definition of acoustic impedance, viz., $Z = p/\dot{X}$, it is clear that a direct measurement of p and $\dot{X}$ will yield Z, or at any rate its absolute value. Such direct measurement has been carried out by Richardson[2] in work on the amplitude of sound waves in resonators. Thus the displacement amplitude at the mouth of a Helmholtz resonator of cylindrical form is measured by the hot wire microphone (to be discussed in Section 8·4), while the excess pressure in the resonator chamber is measured by means of a manometric capsule[3] attached to the back of the resonator. If the radiation

[1] This is not shown in Figs. 8·2 and 8·3. But see the corresponding horn diagrams in Figs. 6·7, 6·10 and 6·12.

[2] E. G. Richardson, Proc. Phys. Soc. London, **40**, 206, 1928.

[3] See E. G. Richardson, Sound, 1927, p. 178.

impedance of the orifice is neglected compared with the inertance, we have then simply for the real part

$$Z_0 = p/\pi a^2 \omega A_0, \tag{8·5}$$

where a is the radius of the orifice, A_0 is the displacement amplitude and p is the pressure, assumed constant throughout the resonator. Note that the velocity amplitude is ω times the displacement amplitude. Actual measurements on a resonator orifice gave $Z_0 = 0.31$ as compared with the theoretically computed value of 0.25 from $\rho_0\omega/c_0$, where c_0 is put equal to $2a$. Further details concerning the measurement of Z, when viscous resistance is accounted for, will be found in the paper to which reference has been made.

A method of measuring the acoustic impedance of porous materials is developed in an article by Wente and Bedell.[1] Suppose we have a cylindrical tube with one end terminated by the material whose acoustic impedance is desired, while the other is attached to a vibrating diaphragm (see Fig. 8·4). Thus in the figure the dia-

FIG. 8·4.

phragm is at O and the material to be tested at O'. The length of the cylinder is l. We can write the expression for the pressure at any point distant x from O as follows:

$$p = \frac{\rho_0 c}{S}\dot{X}_0 \left[\frac{Z_l \cos kl + i\frac{\rho_0 c}{S}\sin kl}{\frac{\rho_0 c}{S}\cos kl + iZ_l \sin kl} \cdot \cos kx - i \sin kx \right], \tag{8·6}$$

where $\dot{X}_0$ = volume current at O, S = the area of cross section of the tube, and Z_l = impedance at O', that is, the impedance to be measured. It is to be noted that the above pressure expression is easily deducible from eq. (5·30) of Section 5·4, where the impedance theory of a cylindrical tube is thoroughly developed. It is also to be noted that viscosity dissipation along the tube is neglected.

[1] E. C. Wente and E. H. Bedell, Bell System Technical Journal, 8, 1, 1928.

The authors suggest three practicable methods of obtaining Z_l by the use of the above equation: (1) by measuring the pressure in phase and magnitude at two points in the tube, (2) by measuring the maximum and minimum pressures at different points along the tube, the length being maintained constant, and (3) by measuring the maximum and minimum values of the pressure at one point, e.g., the source, while the length of the tube is varied. The results quoted in their article are actually obtained by the third method. They are naturally more interested in the absorption coefficient, which is closely related to the impedance.

8·4. Measurement of Sound Intensity.—Resonance Methods.— The classical method for the measurement of sound intensity is undoubtedly that employing the Rayleigh disc.[1] The fundamental fact is that a light disc suspended in a tube through which sound waves are passing tends to set itself so that its plane is perpendicular to the direction of particle displacement in the wave. It is further found that if the disc is suspended so as to lie in equilibrium at a definite angle with the axis of the unexcited tube, the angle through which it turns on the passage of the waves is proportional to the intensity of the sound. König [2] showed that if the original angle of repose of the normal to the disc with respect to the axis of the tube is θ and $\overline{\dot{\xi}^2}$ is the average value of the square of the particle velocity, the moment tending to decrease θ, when the sound passes, is given by

$$M = \frac{4}{3}\rho_0 a^3 \overline{\dot{\xi}^2} \sin 2\theta, \qquad (8\cdot7)$$

if a is the radius of the disc and ρ_0 the density of the medium. It is clear that this is a maximum for $\theta = 45°$, thus indicating the optimum setting. This instrument works best at resonance when its sensitivity is greatest, and for that reason may be called a resonance instrument. The moment M can be measured by means of a torsion suspension and $\overline{\dot{\xi}^2}$ is at once a measure of the intensity of the sound.[3]

[1] Rayleigh, Vol. II, p. 44. See, also, Richardson, Sound, p. 215.

[2] König, Ann. der Physik, **43**, 43, 1891.

[3] Recently a modification of the classical method of using the Rayleigh disc has been proposed by L. J. Sivian (Phil. Mag., **5**, 615, 1928). Instead of measuring the steady deflection in a steady sound field, he modulates the sound amplitude with a frequency equal to that of the free vibration of the suspended disc and then measures

Another resonance instrument is the phonometer of Webster.[1] It consists of a cylindrical resonator tunable by length variation over a wide frequency range with a tuned diaphragm mounted in the resonator opening. The diaphragm is tuned to the sound whose intensity is to be measured by varying the tension in the wires supporting it. A small concave mirror is caused to rotate by the diaphragm motion. Light from a lamp filament is reflected from the mirror into a small telescope and the width of the band of light observed in the micrometer eye-piece measures the diaphragm motion. The pressure amplitude of the sound at the diaphragm is proportional to the displacement amplitude of the diaphragm. The proportionality factor can be calculated from the dimensions of the various parts and the measurements translated into absolute units.

The hot-wire microphone of Tucker and Paris[2] has been extensively used in recent measurements (recall the work of Richardson described in Sec. 8·3). If a fine platinum wire is heated to red heat and exposed to the alternating current of air in the neck of a resonator, there is a falling off in resistance which can be used as a measure of the amplitude of the sound vibrations. For small particle velocities the steady drop in resistance is found to be proportional to $\overline{\dot{\xi}^2}$, and hence to the intensity of the sound.

It is evident that the general type of resonance instrument consists of a small sensitive object (i.e., diaphragm, disc, hot-wire, etc.) placed at the mouth of a resonator. This object must be such that the fluctuating air current at the mouth will markedly disturb its state of motion or physical properties. The change of state can then be observed in a variety of ways, i.e., mechanically, optically, electromagnetically, etc.

8·5. Measurement of Sound Intensity.—Comparison Methods.—If it is desired to measure the intensity of sound produced in the neighborhood of a given source, say an electrically operated loud speaker, it is possible to do so by allowing the current that actuates the speaker to pass through one coil of a mutual inductance which is coupled to a secondary, the e.m.f. generated in which can produce a deflection in a galvanometer via an amplifying circuit. This circuit is also arranged to measure the e.m.f. induced in a telephone

the amplitude of the oscillations corresponding to the modulating frequency. In many practical cases this leads to a gain in absolute sensitivity.

[1] A. G. Webster, Proc. Nat. Acad. Sci., **5**, 173, 1919.

[2] Tucker and E. T. Paris, Roy. Soc. Phil. Trans., **221**, 389, 1921.

receiver (magnetophone) placed in the sound field of the loud speaker. The plan followed is to take a galvanometer reading when the receiver is in the circuit, and then to adjust the mutual inductance of the two coils, so that with the receiver out of the circuit the galvanometer reading is the same as before. The e.m.f. induced in the receiver is then the same as that induced in the secondary coil and, since the former is proportional to the square root of the sound intensity at the receiver and the latter is proportional to the current actuating the loud speaker and the mutual inductance of the two coils, it is possible to express the intensity in terms of this current and inductance and so measure it. This method is a *comparison* method, since it involves the comparison of two e.m.f.'s, one of which is produced by acoustic transformation and the other by purely electrical transformation from the same source.[1] A somewhat similar method uses ear comparison.[2] A known fraction of the current actuating the source is used to operate a receiver connected to the ear. The amount of current in this receiver is adjusted until the sound from it appears (to the ear) to be of equal intensity to the sound from the source. The intensity is then proportional to the square of the current in the receiver. This method is obviously limited to measuring relative intensities. It has been applied to measuring the relative transmission of wave filters (see paper above cited), and the measurement of acoustic impedance described in Section 8·2 is based on a modification of it.

8·6. Pressure Measurements.—For the older absolute measurements of Altberg, Raps, Dvorak and others the reader is referred to the literature,[3] as the details are too numerous to discuss here. Among more recent measurements, however, should be mentioned the interesting work of Barus,[4] who discovered that if a narrow tube with a pin-hole orifice (generally in the shape of a cone) is inserted in a vibratory air column, while its other end is connected to one arm of a sensitive manometer, a measurable static pressure difference (e.g., as much as 3×10^{-4} atmosphere) is recorded when the pin-

[1] For a more detailed account of measurements made with this method see J. C. Karcher, Sci. Papers, Bureau of Standards, No. 473.

[2] See G. W. Stewart, Phys. Rev., **20**, 543, 1922.

[3] For a good resumé consult the *Handbuch der Physik*, Berlin, 1927, Vol. VIII. Akustik, p. 572 ff. The approximate equivalent in English will be found in the recent book *Sound* by E. G. Richardson, London, 1927, p. 218 ff.

[4] Science, **53**, 489, 1921; **65**, 329, 1927. Also: *Acoustic Experiments with the Pin-Hole Probe* and *The Interferometer U-gauge*. Carnegie Institution of Washington, 1927.

hole is at a displacement node (or pressure loop). This provides a direct experimental method of exploring the pressure distribution in a sound field. For a manometer, Barus uses his interferometer U-gauge in which the depression of the mercury surface due to the acoustic pressure is measured optically by interferometric means. Sabine [1] has used the pin-hole probe to explore the sound field in a closed space like a room. Though the action of the probe is probably associated with vortex motions, the theory is unfortunately not completely understood.

In connection with the measurement of sound intensity by means of pressure, attention may be called to the recent *null* method of Gerlach,[2] who compensates the pressure on a diaphragm due to the incident sound wave by measurable electrodynamic forces, so that the diaphragm remains at rest under the two effects. The magnitude of the electrodynamic effect necessary to bring this about is then a measure (when translated into mechanical units) of the sound pressure and hence the intensity.

A more recent method resembling somewhat the compensation scheme of Gerlach is that of Smith [3] who has devised a plan of measuring sound intensity, particularly under water. The essential idea is to obtain a balance between the effect produced on a sensitive amplifying circuit (connected with a receiver) by the sound wave to be measured and the effect of a known small e.m.f. of the same frequency as the sound. The receiver used is of the moving coil type and the effect considered is the deflection of a vibration galvanometer. When the balance is attained, the relation of the force F exerted by the sound wave on the receiver and the balancing e.m.f. E is

$$F = EZ/Hl,$$

where Z is the *total* mechanical impedance of the receiver for the given frequency, and H is the intensity of the magnetic field in which the coil with total length of wire l moves.

8·7. The Condenser Transmitter and the Ribbon Microphone.— The former instrument is thoroughly described by E. C. Wente.[4] Its essential features are briefly indicated in the following diagram (Fig.

[1] Phys. Rev., **23**, 116, 1924.

[2] Wiss. Veröfft. der Siemens-Konzern, **3**, 139, 1923. An account is also to be found in Richardson, loc. cit., p. 220.

[3] F. D. Smith, Proc. Phys. Soc. of London, **41**, 487, 1929.

[4] E. C. Wente, Phys. Rev., **10**, 39, 1917; **19**, 498, 1922.

8·5). The diagram MM' is a thin stretched membrane which is attracted towards a metal "damping" plate P by means of a static charge. The form assumed by the membrane is nearly paraboloidal. Sound waves impinging on the diaphragm vary the capacity of the system, thereby producing a small alternating current in the circuit of which MM' and P are a part. This current must be amplified to be measurable. The stiffness of the membrane is made great enough, so that the first natural frequency is very high, leaving a wide range of frequencies throughout which it will vibrate in its fundamental mode. The region AA' is a very thin film (of order 10^{-3} cm thick), contributing materially to the stiffness and damping of the membrane. For low frequency vibrations the air escaping via A and A' provides increased viscous dissipation, while for high frequencies the air does not have time to escape, so to speak, and the accumulated pressure increases the stiffness of the system. Hence we may say roughly that we have here a system whose stiffness increases with the frequency, while the damping resistance coefficient decreases with the frequency. The total stiffness is, of course, the sum of the intrinsic mechanical stiffness of the diaphragm and the extra stiffness due to the air film. The same is true of the resistance. Now the absolute value of the impedance of the system (see Sec. 2·2) is

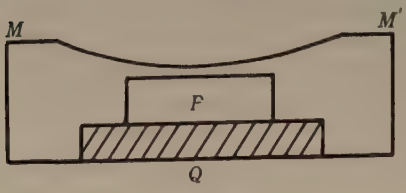

Fig. 8·5.

$$\sqrt{(R_i + R_a)^2 + \left(m\omega - \frac{f_i + f_a}{\omega}\right)^2},$$

where R_i and R_a are the intrinsic and additional resistances, respectively, and f_i and f_a are the corresponding stiffness coefficients. For frequencies far below the first resonance frequency, this reduces to

$$\sqrt{(R_i + R_a)^2 + \frac{(f_i + f_a)^2}{\omega^2}}.$$

From the way R_a and f_a change with the frequency it is found that the change in impedance (over the low-frequency range at any rate) is not so marked as it would be for a system with constant R and f. This is well brought out in the calibration curve of a typical instru-

ment shown in Fig. 8·6.[1] Actually this represents the mean curve for eight instruments very nearly alike. It is to be noticed that the sensitivity as measured by the change in potential in millivolts per dyne per cm² excess pressure is reasonably uniform over a range of frequencies from about 500 to 5000 cycles and changes only by 100% over a range twice as great. When used with an amplifier whose sensitivity decreases appropriately with the frequency, it is seen that a system is available possessing a remarkably uniform sensitivity over a wide range of frequencies. The use of amplification is essential. A more elaborate diagram showing the actual construction of a condenser transmitter is shown in Fig. 8·7.

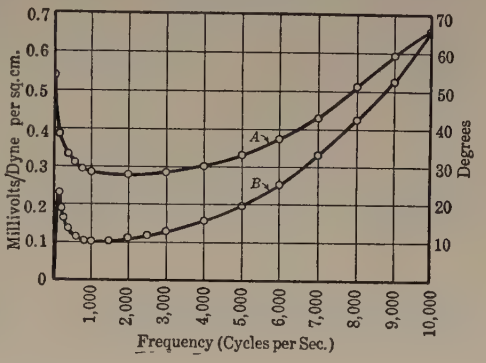

Fig. 8·6.

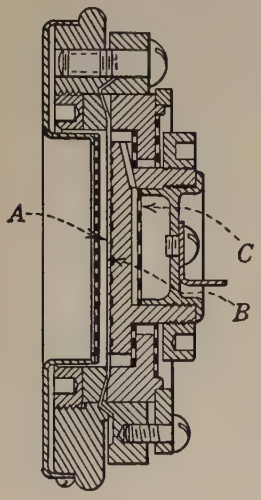

Fig. 8·7.

An electrodynamic instrument of great sensitivity is the ribbon or band microphone of Gerlach and Schottky,[2] which consists of a light metallic (e.g. aluminum) ribbon suspended in a strong magnetic field. The vibration of the ribbon due to an incident sound wave leads to the induction of an e.m.f. corresponding to the undulations of the wave. The ribbon is driven from its equilibrium position by the difference in pressure existing between the two sides. It is so thin that its mechanical impedance can be made comparable to the acoustical impedance of the sound wave in the air at around 4,000 cycles. Hence for frequencies below this value the ribbon will follow very closely the motion of the air particles in the sound

[1] This and Fig. 8·7 are taken from E. C. Wente, Phys. Rev., 19, 498, 1922.

[2] E. Gerlach, Phys. Zs., 25, 675, 1924, W. Schottky, Phys. Zs., 25, 672, 1924.

wave. This instrument requires amplification, but is now being made commercially and is in actual use.

8·8. Membranes and Diaphragms.—Most sound generators of practical use involve the vibration of a membrane or diaphragm and it is therefore of importance to review briefly the salient facts about these vibrators. The theory of the vibrations of a stretched membrane will be found in Rayleigh [1] and will not be given here in detail. We are mainly interested in the practical application of the vibrations of the circular membrane or diaphragm. Moreover, it is essential to note that few diaphragms used in practice are really membranes in the strict sense of the word but are rather plates. It is therefore desirable to emphasize the differences between the vibrations in the two cases.

Considering first the free vibrations of a circular membrane, it is well known that the general differential equation for the normal free vibration displacement ξ, which is

$$\nabla^2 \xi = \frac{1}{c^2} \ddot{\xi}, \tag{8·8}$$

reduces for this case to the form

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \xi}{\partial \theta^2} = \frac{1}{c^2} \ddot{\xi}, \tag{8·9}$$

one solution for which in terms of Bessel's functions is for the case where ξ is a harmonic function of the time [2]

$$\xi = A J_n(kr) \cos n(\theta + \alpha) \cos (\omega t + \epsilon), \tag{8·10}$$

where $k = \omega/c$, r is the distance of the point whose vibration is being considered from the center of the membrane, n is an integer, θ is the angle r makes with a given polar line, A, α and ϵ are constants, and c = velocity = $\sqrt{T/\rho}$, where T is the tension and ρ the surface density. If the membrane is fixed at the periphery, where $r = a$, the radius, we have the additional boundary condition that

$$J_n(ka) = 0, \tag{8·11}$$

and this serves to fix the allowed values of k and hence the natural frequencies of the membrane. The complete solution of the eq. (8·9) will then be formed by summing ξ for all values of n and k.

[1] See Rayleigh, Sound, Vol. I, p. 306. See also Lamb, Dyn. Theory of Sound, p. 141.
[2] See Rayleigh, Vol. I, § 201.

For example, for the lowest mode of vibration, for which $n = 0$, corresponding to motion symmetrical with respect to the center, we have for the roots of (8·11) in order of magnitude

$$ka = .7666\pi, \quad 1.757\pi, \quad 2.755\pi, \quad \cdots,$$

thus fixing the corresponding frequencies at

$$\nu = .766\frac{c}{2a}, \quad 1.757\frac{c}{2a}, \quad 2.755\frac{c}{2a}. \tag{8·12}$$

As above stated, the velocity c is a constant quantity and is dependent solely on the tension in the surface of the membrane (tension per unit length on the surface) and the surface density (mass per unit area).

In Section 2·2 it was found that the resonance frequency of a vibrating membrane treated as an ideal moving piston is given by

$$\nu = \frac{1}{2\pi}\sqrt{f/m},$$

where f and m denote stiffness and mass, respectively. It was there explained that m is, however, in this case not the total mass of the membrane but only the effective mass. By forming the integral for the kinetic energy amplitude, viz.,

$$T = \frac{1}{2}\int \dot{\xi}^2 dm, \tag{8·13}$$

and substituting for $\dot{\xi}$ from (8·10), taking the simple case for which $n = 0$ and using the first approximation whereby

$$J_0(kr) = 1 - \frac{k^2r^2}{4}, \tag{8·14}$$

i.e., the membrane has an approximately paraboloidal shape, it is found that

$$T = \frac{1}{2}\left(\frac{\pi a^2 \rho}{3}\right)\dot{\xi}_0^2, \tag{8·15}$$

where $\dot{\xi}_0$ is the velocity amplitude at the center of the membrane. It is clear, then, that the effective mass is approximately one-third of the total mass. In similar fashion if we calculate the total potential energy amplitude,[1] viz.,

[1] See Rayleigh, loc. cit., Vol. I, § 194.

$$V = \frac{1}{2}\int_0^a \left(\frac{d\xi}{dr}\right)^2 \cdot 2\pi r T dr, \tag{8·16}$$

using the same approximation, we arrive at

$$V = \frac{1}{2}(2\pi T)\xi_0^2, \tag{8·17}$$

so that the effective stiffness is $2\pi T$. And indeed if we now determine the fundamental frequency

$$\nu = \frac{1}{2\pi}\sqrt{f/m},$$

we find

$$\nu = \frac{1.22}{\pi a}\sqrt{\frac{T}{\rho}} = .776\frac{c}{2a}, \tag{8·18}$$

which differs from the first of the frequencies (8·12) by only slightly more than 1%. So far as vibration in the fundamental mode is concerned, it is then possible to replace the membrane by an equivalent piston with mass one-third of the total mass of the membrane and stiffness equal to the tension of the membrane multiplied by 2π.

Examination of (8·10) shows that there are present on the membrane certain *nodal lines*, where no motion takes place. These are of two kinds, concentric *nodal circles* with radii given by the equation

$$J_n(kr) = 0 \tag{8·19}$$

and *nodal diameters* given by

$$\theta + \alpha = \frac{(2m+1)\pi}{2n}, \tag{8·20}$$

where m is an integer. These diameters are n in number and are ranged uniformly about the center, their position in other respects being arbitrary. The following figure (Fig. 8·8) indicates the nature of the nodal diameters and circles for a few special cases. Each diagram is accompanied by the appropriate pair of values of n and s, where s is the number of nodal circles including the circumference. The significance of s is seen when we consider that, if the roots of

$$J_n(ka) = J_n(z) = 0 \tag{8·21}$$

are arranged in order as

$$z_n^{(1)}, \quad z_n^{(2)} \cdots, z_n^{(s)} \cdots,$$

the radii of the corresponding nodal circles will be [1]

$$a, \quad a \cdot z_n^{(1)}/z_n^{(s)}, \quad a \cdot z_n^{(2)}/z_n^{(s)}, \quad \cdots \quad a \cdot z_n^{(s-1)}/z_n^{(s)}.$$

The presence of these nodal circles indicates the existence of radial standing waves traveling with velocity c. It might be questioned whether the wave length of these waves bears any simple relationship to the radii of the nodal circles. As a matter of fact examination of the roots of (8·19) does disclose that for small values of n the *differences* between the radii of successive nodal circles are approximately equal to each other and to a half wave length of the radial wave concerned. (See Fig. 8·8.) For given n this approxi-

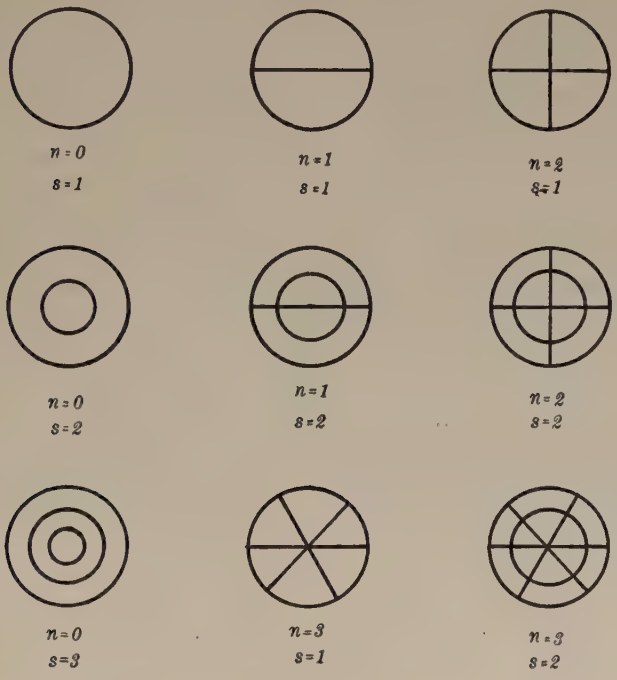

Fig. 8·8.

mation improves with increasing order of the roots of (8·19). On the other hand the existence of the nodal diameters for $n \neq 0$ indicates the presence of *circumferential* standing waves bearing no simple relation to the former and where the nodal lines no longer

[1] See Rayleigh, loc. cit.

have the same simple significance. This is an illustration of the fact that the relation between wave length and distances between nodes which appears in plane stationary waves is not of general application. The consideration of the division of waves into radial and circumferential waves is merely for the purpose of clearness and is not of course strictly accurate. A perfect membrane, without any irregularities whatever, might have solely cylindrical waves. But any lack of complete uniformity would immediately cause the waves to travel in every conceivable direction. On the other hand solely circumferential waves are not at all possible, as a simple consideration of the spreading of waves would indicate.

The number of natural frequencies of a circular membrane is relatively dense in any frequency range above the fundamental. Thus in the first three octaves above the fundamental there are forty-four natural frequencies.[1] This means marked resonance action over a wide range.

With regard to the forced oscillations of a circular membrane the usual resonance phenomena are in evidence. To avoid the nodal lines associated with the overtones, it is often customary in practice (as for example in the condenser transmitter) to drive the membrane at frequencies below its fundamental. The latter can be made as high as necessary by increasing the stiffness.

The influence of the medium on the vibrations of the membrane will be treated in a later section (see Sec. 10·6).

Let us now consider the vibrations of a thin plate. Here again we confine ourselves to the circular form. The fundamental differential equation of motion for free vibrations, deduced on the assumption that the middle layer of the plate is physically inextensible, becomes [2]

$$\nabla^4\xi - k'^4\xi = 0, \tag{8·22}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{8·23}$$

and

$$k'^4 = \omega^2/c^4, \tag{8·24}$$

where

$$c^4 = \frac{qh^2}{3\rho_0(1-\mu^2)}, \tag{8·25}$$

[1] See Handbuch der Physik, Berlin, 1927, Vol. VIII, Akustik, p. 227.

[2] Rayleigh, loc. cit., § 217. See also § 214.

in which q is Young's modulus, $2h$ is the thickness of the plate, μ is Poisson's ratio (i.e., the ratio of lateral contraction to longitudinal elongation), and ρ_0 is the volume density of the plate. The eq. (8·22) may be written in the operator form

$$(\nabla^2 - k'^2)(\nabla^2 + k'^2)\xi = 0, \tag{8·26}$$

and its most general solution is obtained by adding together with arbitrary constants prefixed the general solutions of the equations

$$(\nabla^2 - k'^2)\xi = 0 \tag{8·27}$$

and

$$(\nabla^2 + k'^2)\xi = 0. \tag{8·28}$$

Now the second of the above equations, for the case where ξ is a harmonic function of the time, is of the same form as the eq. (8·8) for the vibrations of a membrane in which the velocity is given by

$$c' = \omega/k'.$$

But the former of the two equations is not a wave equation at all. Now the complete solution is the *sum* of the two separate solutions of (8·27) and (8·28). Hence we are forbidden to think of the presence of waves in the circular plate moving with definite velocity. Nevertheless there is a formal solution of (8·26) as given by Rayleigh [1] which can be put into the form

$$\xi = A \cos(n\theta - \alpha)\{J_n(kr) + \lambda J_n(ikr)\} \cos(\omega t - \epsilon), \tag{8·29}$$

where λ and k must be determined by the boundary conditions. This leads to the conclusion that there exist n nodal diameters, as in the membrane, given by the equation

$$\cos(n\theta - \alpha) = 0,$$

as well as the concentric nodal circles with radii given by

$$J_n(kr) + \lambda J_n(ikr) = 0.$$

The determination of the natural frequencies is difficult. Kirchhoff showed [2] that for frequencies for which the product ka (where a is the radius of the plate) is large there follows, for a *free* edge,

$$ka = \tfrac{1}{2}\pi(n + 2n'),$$

[1] Loc. cit., § 218.

[2] See Rayleigh, loc. cit., § 219.

where n' is the number of nodal circles. But we must emphasize that the actual vibrations in the plate are far more complicated than the simple nodal lines would indicate. Moreover even slight asymmetry of the plate causes the nodal system to move.

The telephone diaphragm is a thin circular plate clamped at the periphery and driven by a periodic force. The symmetrical vibrations are discussed by Rayleigh [1] following the theory outlined above. As a matter of fact, in modern technology it is customary to replace the diaphragm by an equivalent "piston" vibrator, all points of which move together, i.e., have the same displacement at any instant. This is, of course, justifiable only when the diaphragm vibrates below its fundamental, so that all points move in the same phase. In carrying out this replacement it is of course necessary to ascertain the effective mass and stiffness of the piston vibrator, as is discussed in the earlier part of this section for the membrane. This is done by determining the mean velocity over the surface of the plate. Such a calculation [2] shows that for the lowest symmetrical mode of vibration $\bar{\dot{\xi}} = .306\dot{\xi}_0$, where $\dot{\xi}_0$ is the maximum velocity at the center. We shall find it convenient to use the equivalent piston concept freely in connection with the use of diaphragms as sound generators for signalling purposes (Chap. X).

8·9. Modern Loud Speakers.—For many years the telephone receiver was the only practical device for transforming electrical oscillations into mechanical ones. Today the same problem of giving to the air mechanical oscillations corresponding to available electrical oscillations is receiving much attention in the radio industry. Essentially the problem of the transmission of mechanical vibrations to the air is very old. It is difficult to convey the vibrational energy of a solid body to the air, because an amplitude in the air equal to that in a solid involves but a relatively small amount of energy. Success was obtained centuries ago by the use of sounding boards such as are found in all non-wind musical instruments. Within the past century the stethoscope principle was applied in the sound box of the telephone receiver and finally in the phonograph and similar instruments. Elsewhere we have discussed (see Sec. 6·9) the "loading" of a diaphragm by means of a sound box and horn. We desire to give here a more extended discussion of the cone-type loud speaker which is so common in the industry today.

[1] Loc. cit., § 221*a*. Also see Crandall, loc. cit., p. 37.

[2] For details refer to Crandall, loc. cit., p. 37.

There are several important factors involved in the modern loud-speaking art. They may be enumerated as follows: (1) the reception and amplification of electrical oscillations, with retention of relative magnitudes of the amplitudes of the various frequencies of interest; (2) the transformation of these amplified electrical oscillations into mechanical oscillations, again without distortion; (3) the distortionless coupling of the electrically actuated mechanical element to the acoustic radiator or generator, and (4) the effective generation of the aerial acoustic waves.

In each one of the four factors named in the preceding paragraph there is opportunity for serious distortion. It is not desired to emphasize here the electrical aspects, but attention should be drawn to the fact that, even though the moving magnetic or electrodynamic element of the loud speaker operates in a field that is ideal for the purpose of generating the mechanical oscillations without distortion, yet the nature of the mechanism including the magnetic vibrator, the connection to the aerial sound generator and the generator itself, determine whether or not the magnetic vibrator will follow the changing magnetic field in the desired manner. This is because the parts mentioned form a coupling between the magnetic armature or moving coil element and the air itself. As has been shown in the chapter on acoustic wave filters (see Sec. 7·11), we may have a filter in solids and liquids as well as in gases. Any distribution whatever of inertia and elasticity in a mechanical system will cause that system to respond differently to oscillating forces of different frequencies. In the aerial acoustic generator itself, there is distortion. Previous portions of the text (see Secs. 6·1 to 6·6) have described the resonance frequencies of a horn or trumpet. With short horns and frequencies in the acoustic range of speech and music, the distortion caused by resonance is very marked. As the horns are made longer, this distortion becomes less and less noticeable.

At the present time the cone loud speaker is the most common one in use. It is actuated at its vertex and the treatment of the base of the cone varies in different instruments. The discussion here will be confined to the case of the cone with a free edge. An understanding of its manner of vibration can best be understood from a study of the vibration of a thin circular plate, unclamped and actuated by vibrations conveyed to its center. We have already discussed this in the previous section (Sec. 8·8) and may assume an

understanding of the ideas presented there. The natural frequencies of the cone may be studied in a manner analogous to that pursued with the plate. Several new considerations arise, however, that are worthy of mention.

First, while the vibrations of the flat plate are accompanied by relatively slight extensions in the material, such vibrations normal to the surface in a curved plate may involve relatively large extensions. This fact renders a curved surface, such as a cone, more rigid to flexural vibrations and consequently not so easily broken up into nodes and loops. Second, if a curved plate is sufficiently thin, flexural vibrations without extension may be the most important type of vibration. This is appreciated if we consider the flexural vibrations in a thin cylindrical plate. Oscillations may occur which permit the length of the circumference to remain constant and the form of the circumference to oscillate with the circle as its mean. If we draw two oblong somewhat elliptical forms representing the two extreme positions taken by the circumference, we note that the simplest type of such a vibration involves four nodes and that at these nodes there is movement tangential to the circle. Hence, there may be purely flexural movements with no extension. This type of movement may be expected to be important in the thin conical plate.

From the above we might expect the cone to have natural vibrations with nodes and loops which, in projection, much resemble the patterns in a flat plate, with the nodal lines generators of the cone and circles having points on the axis as centers. In order to have flexure without extension the number of generatrix nodes is even. These anticipations are borne out by experiment.

In a loud speaker the cone is actuated at the vertex with a movement that is axial. The motion thus conveyed to the cone may be separated into components, one normal to the surface and one in a generator of the cone. The former would consist of flexural waves accompanied by extension and the latter of extensional waves only. Because of the rigidity of shape, a cone of the same thickness will have a higher fundamental natural frequency and hence higher corresponding natural frequencies than a flat plate. Within the acoustic range, therefore, there would be fewer resonance frequencies in the cone. But because of resonance the cone will not convey to the air the same relative amplitudes of various frequencies as in the vibratory movement given the apex. Dis-

tortion will occur. To avoid this in part, electrical wave filters are frequently employed to reduce the magnitude of the amplitude of the electrical oscillations.[1] Obviously the shape of the generator does not need to be conical. Some are conoidal.[2] There are various methods by means of which the conical acoustic generator may be used. There may be a double cone, partially enclosing a volume of air which has a loading effect. The base of the cone may be supported by thin leather or rubber; it may vibrate between felt, or it may have felt attached thereto, or it may be mounted in a ring diaphragm. The selecting of a method may be determined by trial and by the cost of manufacture. The shape and extension at the edge of the cone are important because this surface forms a "baffle." It not only acts as a reflector, but it assists in preventing the air slippage from one side of the cone to the other. This slippage reduces the pressure that would otherwise be developed by the vibrating cone. Obviously its effect is more serious at low frequencies. The difficulties involved in making a distortionless loud speaker are inherent, as has been pointed out, and they can be eliminated generally, not by direct removal, but by the introduction of appropriate devices, such as electrical and mechanical filters. But even if a distortionless loud speaker *were* constructed, it would need to be operated so as to give the original acoustic intensity. This statement is based upon the fact that the drumskin of the ear is an asymmetrical vibrator (see Sec. 9·8) which magnifies the difference tones and hence the low frequencies. If the actual intensities of all frequencies in a complex sound were equally increased, the ear would increase the low frequencies relatively more than the others. Hence the ear will, in general, introduce distortion. This leads to the point that any acoustic vibrating diaphragm will introduce frequencies other than those imparted and for several reasons. It ceases to be a symmetrical vibrator when in actual use. Again, if the vibrations are too large, it will rattle. Presumably, the latter is caused not only by looseness of parts in the mechanism, but also by the extreme distortions of the diaphragm itself. A bottom of an oil can will snap from one stable position to another, giving off characteristic sounds of the material. It seems that no study of the

[1] Resonance effects may be reduced by the use of felt at the base of the cone and by the viscosity of the material of which the curved surface is made.

[2] For a more elaborate discussion of the vibration of curved plates in general see Rayleigh, loc. cit., Vol. I, 1926, p. 395.

extraneous noises of thin diaphragms has been made and the remarks here made are merely indicative rather than specific.

From what has been stated it is clear that the smaller the cone the less the number of resonance frequencies in the audible range. Thus a cone having a few inches diameter at the base will give good results if sufficient power is used to drive it. The flow of energy to the air will of course depend upon the surface exposed. Also if the cone is very large the resonance frequencies will be so numerous and so distributed as to give good reproduction. But the ideal is to avoid distortion entirely and two efforts in this direction will be mentioned.

At present the condenser loud speaker promises to become practicable. The advantage will be the use of the electrical oscillations directly varying the attraction between two "plates" and causing one to move in response. The object to be attained is the motion of the "plate" as a whole, thus avoiding any resonance distortion. With the absence of a distorting operating mechanism and of resonance, the reproduction should be very good.

Dr. C. W. Hewlett[1] a number of years ago devised a loud speaker in which the diaphragm moves as a whole between two flat coils. His instrument is practically free from distortion, but requires considerable power to operate.

It is not the purpose of this section to give a specific review of loud speakers. For the literature concerning them the reader is referred to the references quoted in Crandall's "Theory of Vibrating Systems," pp. 246–247.

8·10. The Efficiency of Sound Generators.—The acoustic efficiency of a sound generator is the ratio of the acoustic output (that is, the rate of energy flow in the wave train from the generator) to the mechanical input. With the increasing use of sound sources for many practical purposes such as signalling, etc., it has become important to consider ways of measuring this quantity, as well as means of increasing it for given types of generators.

King[2] has studied compressed air generators (i.e., whistles and sirens) and found that the efficiency may be expressed by the simple formula

$$\eta = \frac{T_1 - T}{[1 - (p_0/p_1)^{(\gamma-1)/\gamma}]T_1}, \tag{8·30}$$

[1] Radio Broadcast, **7**, p. 508, 1925.

[2] L. V. King, Phil. Trans. Roy. Soc., *A*218, 223, 1919.

where T and T_1 are the absolute temperatures of the air on the low-pressure and high-pressure sides respectively, p_0 and p_1 are atmospheric pressure and operating pressure respectively, and $\gamma = 1.41$, the ratio of the specific heats for air. The efficiency thus increases with the temperature difference between the two sides and decreases with the operating pressure. All the quantities entering into the expression are easily measurable. For example, the temperature difference may be as high as 5° C. and can be measured by a portable thermopile.

The membrane or diaphragm is probably the most important practical sound generator now in use for low-frequency work. It will be shown in Section 10·6 that, if we consider the diaphragm replaced by an equivalent piston, all points of which have the same displacement velocity, the average rate of radiation of sound from one side of this diaphragm into a semi-infinite medium of density ρ_0 is

$$\tfrac{1}{4}\pi\rho_0 c k^2 a^4 |\dot{\xi}|^2, \qquad (8{\cdot}31)$$

where a = the radius of the equivalent piston and $|\dot{\xi}|^2$ is the square of the maximum displacement velocity. By the use of a formula essentially like the one above, King [1] has measured the efficiencies of telephone receiver diaphragms and has found values ranging from 6×10^{-6} at 400 cycles to 4×10^{-3} at 1000 cycles. The improvement of telephone receivers from the standpoint of efficiency is an important problem.

The efficiency of the Fessenden oscillator described in Section 10·7 and used for subaqueous sound signalling has been rated as high as 50% with an output of 500 watts.

It may be of interest to note that Sabine [2] has measured the acoustic output of various musical instruments. For example, that of a violoncello at 128 cycles was 10^{-4} watt, while at 650 cycles it fell to 10^{-6} watt. A good violin gave out a uniform output of 6×10^{-5} watt over a frequency range from 200 to 1300 cycles. The actual acoustic efficiencies were not measured.

A precision source of sound which has been studied extensively in recent years is the *thermophone*. If alternating current is passed through a very thin conductor, the periodic heating gives rise to periodic temperature changes in the medium near the wire and

[1] L. V. King, Jour. Frank. Inst., Vol. 187, p. 611, 1919.
[2] P. E. Sabine, Phys. Rev., 22, 303, 1923.

sound waves are produced of a frequency double that of the alternating current. This instrument has been investigated by Arnold and Crandall,[1] and more recently by Wente.[2] The acoustic efficiency can be quite readily calculated from the output as measured by a condenser transmitter and the input of electrical energy. It is found to compare favorably with electromagnetic and electrostatic devices except in the vicinity of their resonant frequencies. More important, perhaps, is its ease of adjustment and its constancy of response over long periods of time. Moreover the reactions of the medium on the source, so important in connection with diaphragm sources, are here negligible.

The enormous development of the radio industry has led to considerable research on methods of improving the efficiency of the driving units of loud speakers. Until very recently the average efficiency of such devices has been about 1%. Recent studies at the Bell Telephone Laboratories,[3] however, have led to the development of a diaphragm unit which, when coupled to an "ideal" horn, i.e., one which has at its throat the same acoustic impedance as a tube of infinite length, yields an efficiency of practically 50% over a frequency range from 70 to 4000 cycles. Moreover, there is little variation of efficiency with frequency over this range, which is a condition of much more importance in the quality of reproduction. It is interesting to note that the efficiency of this unit was determined by measuring the acoustic output directly, that is, by observing the excess pressure produced in an equivalent "infinite" tube attached to the throat. The input, of course, is measured electrically.

Questions and Problems

1. Given a circular sheet steel membrane of thickness .005 cm and radius 2 cm clamped about the edge and held with a superficial tension of 5×10^7 dynes/cm. Calculate the first, second and third natural frequencies of the membrane. What is the effect on these frequencies of assuming a damping factor per unit area

$$R_1 = 200 \frac{\text{dynes sec}}{\text{cm}} \frac{1}{\text{cm}^2} ?$$

Calculate the logarithmic decrement or decay modulus for this case.

[1] H. D. Arnold and I. B. Crandall, Phys. Rev., **10**, 22, 1917.
[2] E. C. Wente, Phys. Rev., **19**, 333, 1922.
[3] E. C. Wente and A. L. Thuras, Bell System Technical Journal, **8**, 140, 1928.

2. Develop the theory of the condenser transmitter (see Crandall, p. 29) and emphasize the physical significance of the results.

3. A condenser transmitter system has a disc of effective (piston-like) radius = 1.63 cm. The mean separation distance of disc and damping plate is $D = 2.2 \times 10^{-3}$ cm. Calculate the extra resistance coefficient and the extra stiffness coefficient due to the air damping at the center of the disc. The resistance coefficient of the air is $R = 12\mu/D^2$, where μ = coef. of viscosity = 0.000186 c.g.s. units at 20° C. Take atmospheric pressure as 10^6 dynes/cm^2. Carry through the computation for the frequencies 1000, 2000, 4000 and 6000 cycles.

4. Using the data in question 3, calculate by integration the *average* stiffness and the *average* resistance due to the air damping over the whole disc for the frequencies stated.

5. Discuss the problem of measuring the acoustic impedance of the external ear at the entrance into the meatus. Describe the difficulties involved and the reason for the method you select.

6. Carry through the derivation of eqs. (8·15) and (8·17) of Section 8·8.

CHAPTER IX

Physiological Acoustics [1]

9·1. Energy Flow in Speech.—The energy flow in speech is very far from constant. In a study by Sacia [2] it is shown that the *mean* power for a syllable by a speaker may be of the order of 60 to 120 microwatts, while in the same syllable the *peak* power may be as great as 1000 to 2000 microwatts. Sacia found that the mean power per sq. cm in what might be called normal speech is about 7 microwatts at a point 9 cm from the mouth, when the mean is taken over the whole length of time. In these tests the average ratio of the total time in the silent gaps to that consumed by the syllables is 0.55.

9·2. Energy in Speech Sounds.—The relative powers in arbitrary units were found by Sacia and Beck [3] for a number of vowels and consonants and these are presented in Table I. These indicate that the vowels rank the highest, the semi-vowels next and the consonants the lowest. They show that there is no fixed relationship between mean and peak power. For example, *a* as in *tap* has a peak power that is six times its mean power as given and consequently may be guilty of overloading an electrical circuit adjusted for the mean power in conversation. For the meaning of "articulation" Section 9·4 should be consulted.

9·3. Nature of Speech Sounds.—For many years it was not definitely known whether sustained vowel sounds owe their peculiar quality to the relative pitch of the fundamental and overtones, or to the fixed pitch of the characteristic tones or overtones. Professor D. C. Miller [4] settled this matter most conclusively by showing experimentally that the peculiar quality of a sustained vowel depends to a large extent, upon certain frequency regions, irrespective of the fundamental tone used. The most complete study

[1] The material in this chapter is abbreviated, since a recently published work (Fletcher, *Speech and Hearing;* D. Van Nostrand Co.) covers the field much more adequately than could a book of this compass.

[2] Sacia, Bell System Tech. Jl., IV (1925), p. 627.

[3] Sacia and Beck, Bell System Tech. Jl., V (1926), p. 393.

[4] Miller, Science of Musical Sounds, Macmillan, New York, 1916.

TABLE I

Speech Sounds

Relative Power, Arbitrary Units

Speech Sound	Key	A Mean Power Conversational Values for 16 Speakers	B Peak Power Normal Values for 2 Speakers	C Relative Power Attenuation to Give 80% Articulation
ò	talk	1870	688	826
a	top	1380	1430	474
ō	tone	875	630	619
ā	tape	808	632	567
e	ten	664	975	364
o	ton	616	688	474
ū	tool	532	344	349
ē	teem	484	402	421
r̄	err	384	— see note	924
à	tap	366	2170	645
i	tip	346	688	295
n	no	84	78	36
m	me	74	185	38
sh	shot	73	192	216
ch	chat	58	87	64
s	sit	38	51	11
z	zip	29	52	17
j	jot	19	41	98
ng	ring	14	162	134
k	kit	14	10	43
l	let	13	218	157
t	tap	6	26	32
d	dot	3	7	60
f	for	3	6	9
v	vat	1	41	13
u	took	— see note	688	347
zh	azure	—	63	—
dh	that	—	15	—
g	get	—	13	60
b	bat	—	11	30
p	pot	—	11	24
th	thin	—	1	1

Note: The dash indicates that observations were not available.

yet made of the distribution of energy among the frequencies for the different vowels is that published by Crandall.[1] A vowel is not the same throughout its duration. For example, the vowel $\overline{oo}$ as in pool, in the case of one speaker, required 0.5 sec. to build up.

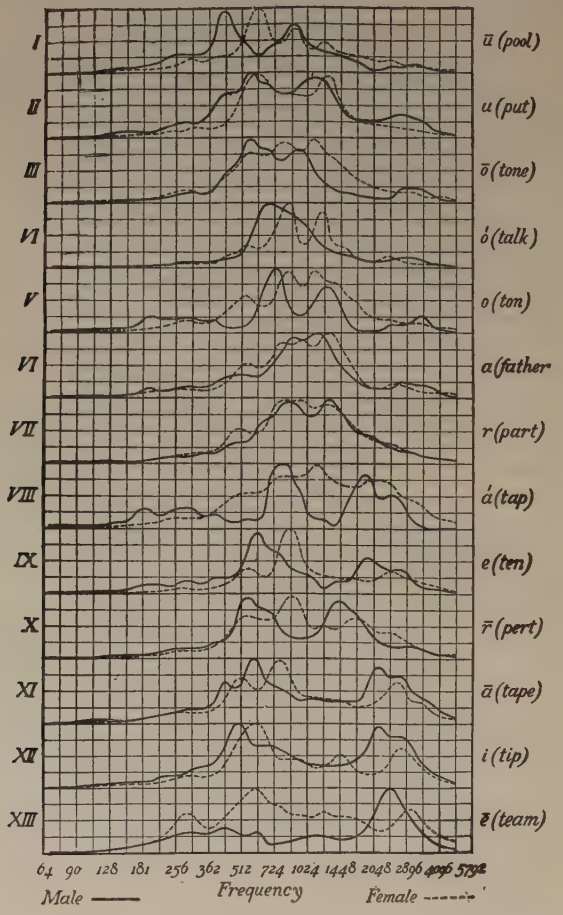

Fig. 9·1.

There was a middle period of 0.20 sec. followed by a period of decay of 0.31 sec. Without giving the details of the analyses of the recorded curves for the vowel sounds in speech, it may be understood

[1] Crandall, Bell System Tech. Jl., IV (1925), p. 586. Figures 9·1 and 9·2 are taken from this source.

that the records for the entire duration of each vowel were used and the component frequencies were ascertained by a mechanical method of analysis. A group of eight records was made of each vowel sound, four with men and four with women. The results are expressed in the form of curves shown in Fig. 9·1. The full line curves are for the male and the dotted for the female. The curves represent the relative importance of the amplitudes at the different frequencies when the variation in sensitivity of the ear with frequency is taken into consideration. It is noted that, in general, the frequency regions of importance are numerous. In *a* as in *father* there is a broad continuous or single region. In *ā*, *i* and ē there are two very distinct regions. The differences between the

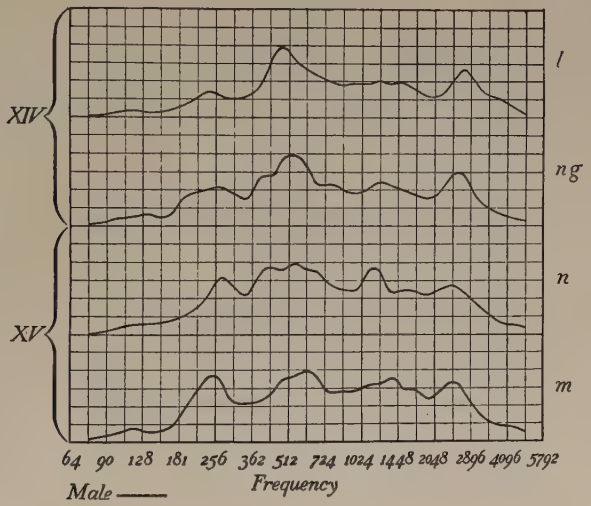

Fig. 9·2.

male and female curves are probably caused by the differences in the fundamental tone upon which the vowels are sounded rather than by differences in the resonance cavities. It is to be borne in mind that the resonances of the larynx, pharynx, mouth and nose are the cause of the large amount of energy obtained from the vocal cords and also account for the ability to make the characteristic sounds of speech. In addition to the eleven vowels, two of the semi-vowels are shown in Fig. 9·1. Four others are given in Fig. 9·2. Fletcher,[1] after considering the work of Stumpf, Miller,

[1] Fletcher, Speech and Hearing, loc. cit., p. 58.

Paget and Crandall, has compiled Table II, showing the characteristic frequency region for the vowel sounds. Table III indicates the frequency regions for the consonants r, l, ng, m and n.

TABLE II

CHARACTERISTIC FREQUENCY OF THE VOWEL SOUNDS

Speech Sound	Low Frequency	High Frequency
u (pool)	400	800
u (put)	475	1000
o (tone)	500	850
a (talk)	600	950
o (ton)	700	1150
a (father)	825	1200
a (tap)	750	1800
e (ten)	550	1900
er (pert)	500	1500
a (tape)	550	2100
i (tip)	450	2200
e (team)	375	2400

TABLE III

Sound	Throat Resonance	Nasal Resonance	Mouth Resonance
r	500–700	1000–1600	1800–2400
l	250–400	600	2000–3000
n	200–250	600	1400–2000
ng	200–250	600	2300–2600
m	250–300	600	900–1700

It should be observed that the foregoing contains merely the barest outline of the physical causes of the differentiation of vowel sounds. A source and resonating chambers exist. Further details would involve not only the influence of resonance chambers on sustained sounds but also on impulsive and rapidly varying sounds. Moreover, the vocal chords are by no means a simple type of reed and the walls of the spaces involved are not rigid and stationary. In fact the production of speech is very complicated and involves acoustical phenomena that are well within the range of the ear, but yet not discernible by it. For example, there is a marked difference between the vowel in *eat* and in *meat*, the higher fre-

quencies sliding upward in the latter at the beginning of the vowel. Yet this change in pitch is not recognized by the ear. What might be termed fairly complete studies of speech must be made by physical measurement and must involve many facts that are of no importance to the air but of great value in securing a desirable background for phonetics.

9·4. Frequencies Important in Speech.—Valuable information has been obtained by Fletcher [1] concerning the relative importance of frequencies in English speech. This importance was determined by the "articulation" or the percentage of syllables understood in selected lists of syllables. The following interesting conclusions may be drawn from Fletcher's experiments:

1. The importance of frequencies higher than 1550 cycles is just as great as that of those below this value.

2. If only frequencies below 1000 cycles are eliminated, the articulation is 86%.

3. If only frequencies below 1000 cycles are used, the articulation is 40%.

4. Eliminating all frequencies below 1000 cycles gives the same articulation as eliminating all those above 3000. The curves showing these and other conclusions are given in Fig. 9·3. [2]

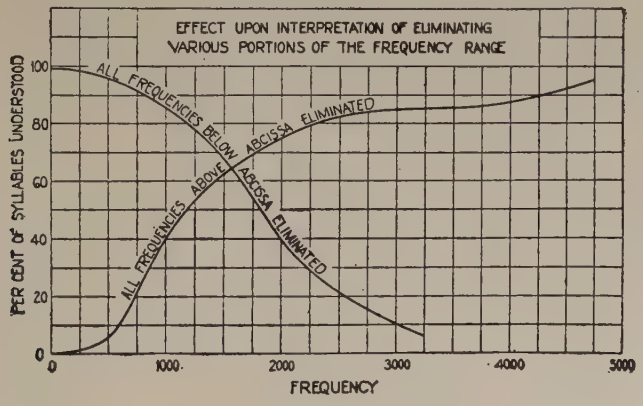

FIG. 9·3.

9·5. Minimum Audibility.—Apparently the most reliable values for the minimum audible pressure in dynes per sq cm are as follows: [3]

[1] Fletcher, Bell Sys. Tech. Jl., I (1922).

[2] This and Fig. 9·4 are taken from Fletcher, Bell Sys. Tech. Jl., 4, 375, 1925.

[3] Fletcher, Bell System Tech. Jl., 4, 375, 1925.

Frequency	64	128	256	512	1024	2048	4096
Threshold pressure in dynes/cm^2.....	0.12	.021	.0039	.001	.00052	.00041	.00042

The values refer to the r.m.s. values of pressure and can be made to yield the flow of energy in ergs per sq. cm per second by the fact that the latter is the square of the former divided by $\rho_0 c$ (see Sec. 1·15) or by approximately 42 in c.g.s. units. If one desires the values in microwatts per sq. cm, E, the transforming equation is $p^2 = 20.5E$. It is observed, then, that the ear is remarkably sensitive, detecting a changing pressure of approximately $\pm \sqrt{2} \times 10^{-3}$ dyne or a flow of energy of about 5×10^{-8} microwatt. If one had a source of sound radiating 1 watt per second and if the flow varied inversely as the square of the distance, then at a distance of 13 km the flow would be approximately 5×10^{-8} microwatt. The values in the above tables, the results of experiments with almost 100 ears, may be considered to refer to the normal ear. It is to be observed that the threshold pressure refers to the pressure at the ear and not in the approximately plane wave in the absence of the head. The head would approximately double the pressure for very short wave lengths and would increase the pressure to some degree even for low frequencies.

9·6. Limits of Audibility.—A sound may be too faint to be heard and it may be intense enough to cause pain. There is then a maximum pressure limit for audibility as well as a minimum. The observations on this maximum have been made only with ears that are approximately normal. With this limitation one can apply the results of Fletcher and Wegel [1] as shown in Fig. 9·4. Half of the observations lie within the dotted curves. The frequency limits of audibility as shown are arbitrary, the variation with different individuals being too wide to have an acceptable norm. As a guide to the relation between common observations of deafness and the values of the r.m.s. pressures in dynes, Fletcher and Wegel [2] state that persons called "slightly deaf" require a pressure variation from the normal of about 0.1 dyne (r.m.s.) per sq. cm, that a person requiring 1.0 dyne (r.m.s.) per sq. cm can usually follow ordinary conversation and that those who require 10 dynes (r.m.s.) per sq. cm

[1] See Fletcher, Bell System Tech. Jl., loc. cit.

[2] Fletcher and Wegel, Phys. Rev., **19**, p. 553 (1922).

require artificial aids to hearing. In this connection it may be said that studies in deafness in various parts of the sound spectrum are being made at various laboratories with the expectation that the results will serve as a means of detecting approaching noticeable deafness and the location of the cause.

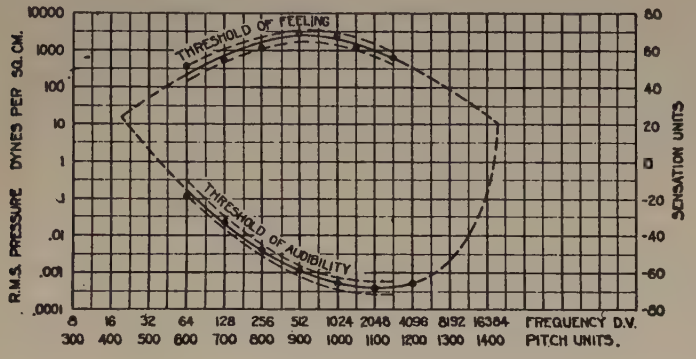

FIG. 9·4.

9·7. Loudness.—Loudness is a sensation and its measurement in terms of physical values is not at all simple. It is a topic discussed at length in psychological literature. Our consideration will be limited to certain recent experiments. MacKenzie[1] has performed experiments on the relative sensitivity of the ear at different levels of loudness and his results are in conformity with the equation

$$S = c \log_{10} p + a,$$

wherein S is the sensation of loudness, p is the excess pressure and c and a are parameters dependent upon the frequency, although c varied not more than 10% over the range of frequencies tested, 100 to 4000. The range of loudness was from the threshold value to 20,000 times that value. From the equation it is obvious that a is a function of the threshold pressure. For loud tones the value of a is small in comparison with $c \log_{10} p$ and the sensation is practically a logarithmic one. This accounts for the fact that the intensity of a sound wave increases at a much greater rapidity than its loudness.

The law expressed in the preceding equation is sometimes called Fechner's Law. MacKenzie does not experiment with the law in

[1] MacKenzie, Phys. Rev., 20, 1922, p. 331.

this form but derives a formula showing the relation of two pressures, or

$$\log_{10} p_1 = \frac{a_2 - a_1}{c_1} + \frac{c_2}{c_1} \log_{10} p_2 = A + B \log_{10} p_2.$$

This formula is verified and experimental values of A and B are given, but not the individual values of a and c for each frequency.

For the sake of convenience several authors are using the terms *sensation unit* and *sensation level.* They may be defined as follows:

$$\text{One sensation unit} = 20 \log_{10} p,$$

$$\text{The sensation level} = 20 \log_{10} \frac{p}{p_0},$$

wherein p_0 is the threshold pressure. The sensation level is really the number of sensation units required to reduce the tone to the threshold limit. In Section 1·15 the decibel as a unit of power was defined so that the number of decibels between the level I_0 (the threshold) and the given level I is

$$\alpha = 10 \log_{10} I/I_0.$$

But the power is proportional to the square of the pressure (see eq. (1·50)) and hence

$$\alpha = 20 \log_{10} p/p_0.$$

Thus the sensation level may be expressed in decibels.

The above discussion of loudness is by no means complete. While MacKenzie's work has been largely superseded by more detailed studies [1] it presents a relatively simple aspect of the phenomena. One of the complications that arises is the loudness of complex sounds. Very interesting experimental contributions have been made but the presentation of the results here would lead too far from the purposes of this book.

9·8. Minimum Perceptible Intensity Difference.—There is for every frequency a minimum perceptible difference in intensity. The most extensive results are those of Knudsen.[2] They may be presented as follows:

[1] See Fletcher, Speech and Hearing, loc. cit.

[2] Knudsen, Phys. Rev., **21**, 84 (1923).

SL	5	10	20	30	40	50	60	70	80	90	100
$\frac{\Delta E}{E}$	0.4	0.23	0.14	0.12	0.11	0.106	0.102	0.10	0.10	0.10	0.10

SL is the sensation level and E is the intensity. Knudsen has also shown that $\Delta E/E$ at any sensation level is practically independent of frequency, varying only 10% from 100 to 3200 cycles. The statement $\Delta E/E$ = a constant is really Fechner's Law. If it were accurate, the logarithmic sensation law could be derived therefrom. It is interesting to note that at 1000 cycles the ear can distinguish about 400 gradations of loudness between the limits of intensity within which the ear responds. The number of gradations decreases above 2000 and below 1000 cycles.

The work of Riesz [1] is fairly in accord with that of Knudsen, but modifies the result that $\Delta E/E$ is independent of frequency. Riesz shows that for frequencies up to 10,000 cycles the latter is approximately correct for sensation levels above 50 decibels. He found that

$$\Delta E/E = S_\infty + (S_0 - S_\infty)(E_0/E)\gamma$$

will represent $\Delta E/E$ as a function of intensity. S_∞ is the value of $\Delta E/E$ at high intensities, S_0 the value at the threshold and γ is a number. These constants, varying with frequency, may be expressed as follows:

$$S = .000015f + 126/(80f^{0.5} + f),$$
$$S_0 = 0.3 + 0.0003f + 193/f^{0.8},$$
$$\gamma = 244000/(358000f^{0.125} + f^2) + 0.65f/(3500 + f),$$

where the frequency is here indicated by f.

9·9. Pitch Differences and Pitch Levels.—According to Knudsen (see preceding reference) the average single ear can just appreciate a difference of 0.5 cycle at 50 cycles, 0.66 at 100 cycles, 1.6 at 500 cycles, 3.0 at 1000 cycles, and 9.0 at 3000 cycles. It has been proposed to adopt a "pitch level" analogous to "intensity level" and "sensation level" earlier mentioned in this work. The expression suggested is

$$P = A \log_2 f,$$

where P is expressed in octaves and f in kilocycles.

[1] Riesz, Phys. Rev., **31**, 867 (1928).

9·10. Summation and Difference Tones.—When two frequencies, n_1 and n_2, are present, several interesting phenomena occur. The two vibrations combine into one of varying intensity, especially noticeable when n_1 and n_2 are nearly equal, and the well known definite "beats" are produced. It is interesting that the mean pressure is not the same throughout the fluctuation of intensity. For Rayleigh showed [1] that a plane wave of sound striking a wall normally will exert a mean excess pressure. This pressure is determined by the energy per unit volume of the incident acoustic wave. In fact it is $2E$, where E is the energy per unit volume. It is this pressure that fluctuates with the frequency $(n_1 - n_2)$. But this pressure, when computed for any ordinary case, proves to be too small to cause any sensation in the ear. Nevertheless the ear can detect a "difference" tone having the frequency $(n_1 - n_2)$. In fact, a summation tone of frequency $(n_1 + n_2)$ can also be heard. But this is not all. Multiples of these combination frequencies may also be heard. The explanation is to be found in the nature of the ear drumskin. It is an asymmetrical vibrator. That is, the displacement for a given pressure variation is not the same in the positive as in the negative direction. In Appendix VI it is shown that an asymmetrical vibrator will, when actuated by a combination of two simple harmonic forces of frequencies n_1 and n_2, vibrate with multiples of these frequencies and also of combination frequencies. Wegel and Lane,[2] using $n_1 = 1200$ and $n_2 = 700$, have heard the following frequencies: n_1, n_2, $n_1 + n_2$, $n_1 - n_2$, $2n_1$, $2n_2$, $3n_1$, $3n_2$, $2n_1 + n_2$, $2n_1 - n_2$, $2n_2 + n_1$, $2n_2 - n_1$, $4n_2$, $2n_1 + 2n_2$, $2n_1 - 2n_2$, $3n_1 + n_2$, $3n_1 - n_2$, $3n_2 + n_1$, and $3n_2 - n_1$. All these can be accounted for by assuming the displacement of the drumskin is expressed in terms of the first, second, third and fourth powers of the actuating pressure. The only frequency predicted by the asymmetry theory and not detected was $4n_1$. As might be expected, the asymmetry of the vibration of the drumskin decreases with decreasing amplitude of vibration.

9·11. Theories of Audition.—Physiological processes should of course be interpreted ultimately in terms of physics and chemistry. But the road to a satisfactory interpretation of audition has been long and the end of the journey is not yet in sight. Fletcher [3] seems

[1] Phil. Mag., **3**, 338, 1902.

[2] Wegel and Lane, Phys. Rev., **23**, 266, 1924.

[3] Speech and Hearing, loc. cit.

to adopt an "extension" of the Helmholtz theory and to accept the existence of resonance of the basilar membrane, though endowing it with much damping. He concludes that experiments of today favor the theory that loudness depends upon the "total number of nervous discharges coming to the brain" and that these are dependent upon "the number of nerve fibers stimulated and the intensity of stimulation of each one." The recognition of pitch, according to Fletcher, depends upon characteristic frequency regions of the basilar membrane. When the maximum stimulation occurs in a certain region of the basilar membrane, the pitch recognized is that corresponding to this region.

A very different view is held by Meyer,[1] who advocates what might be termed the "hydraulic" theory. He has set up an equation "describing the hydraulic functions of the mammalian cochlea." By the use of this equation he is able to show agreement with auditory observation and with experimentation on a large transparent hydraulic model. Meyer would agree with Fletcher that loudness depends upon the number of cells stimulated, but he holds that "the nature of the stimulation and accordingly the quality of the chemical process resulting in each of the sensitive cells" determines the sensation quality or pitch.

More recently [2] two somewhat similar theories of hearing have been presented by L. T. Troland and Harvey Fletcher. Only a brief reference can be made to the fundamental physical difference in view. Troland adopts the time pattern view of the dependence of the sensation of pitch upon the time nature of the stimulation. Fletcher also adopts this view but adheres to the resonance feature or the importance of the position of excitation on the basilar membrane. He calls his contribution a "space-time pattern theory." It seems clear that a thoroughgoing biophysical and biochemical study of the auditory nerve process is imperative. Only a considerable progress in this direction will remove present serious barriers to the further development of theories of hearing.

9·12. Binaural Effects.—It is known to all that the ears can localize the direction of a source of sound to a distinct extent. A source on the right of a plane bisecting and normal to the line joining the ears (hereinafter called the median plane) can be

[1] Max F. Meyer, Jl. of Gen. Psych., 1, 239, 1928, and numerous other publications by the same author therein mentioned.

[2] At a meeting of the Acoustical Society of America, December, 1929.

recognized as on the right. A simple explanation of the ability of the two ears to locate sound in this manner might be that obviously the source of sound appears to be on the side where there is a greater intensity at the ear. This explanation was the one accepted for many years, but must now be discarded. As will be shown, the difference of phase at the two ears is more effective in localizing the origin of the sound. But there *is* an effect caused by an intensity difference at the ears, the phase being kept the same. Its magnitude and its variation with the intensity ratio at the ears have been studied, but not extensively. Stewart and Hovda [1] found that if θ represents the angle made with the median plane by a line drawn in the apparent direction of the sound-source, and if the experimenter produces a sound intensity I_R at the right ear and an intensity I_L at the left ear, both having the same frequency and the same phase, then there is the following relation between the factors involved:

$$\theta = K \log_e \frac{I_L}{I_R},$$

wherein K is a constant. The value of the constant differs [2] with individuals and with the frequency. For example, for an observer B, the values of K for 256, 512 and 1024 cycles were respectively 30°, 21° and 10°. If one considers a source of sound removed at an ordinary distance and computes what differences of intensity would be produced at the ears by the source located at an angle θ from the median plane, it is found (see loc. cit.) that the above formula does not express or explain the phenomenon of localization. In fact, it is found that with observers generally the foregoing "binaural intensity effect" ceases to exist in certain frequency regions wherein there is no cessation of the localization ability.

If the intensities at the ears are maintained alike and differences of phase are introduced, then there is an angular displacement of the apparent sound source or phantom source from the median plane and the following relation obtains between the angular displacement, θ, and the phase difference, φ:

$$\varphi = K_1\theta.$$

Again K_1 varies with frequency, but in a linear manner,[3] as is shown

[1] Stewart and Hovda, Psych. Rev., XXV, 3 (1918), p. 242.
[2] Stewart, Phys. Rev., XV, 5 (1920), p. 425.
[3] Stewart, Phys. Rev., XV, 5 (1920), p. 425.

in Fig. 9·5, which shows the variation in φ/θ or K_1 with frequency for three observers.[1] Two significant things can be shown from these experimental curves. First, the curves are what would be predicted upon the theory that localization depends upon phase.

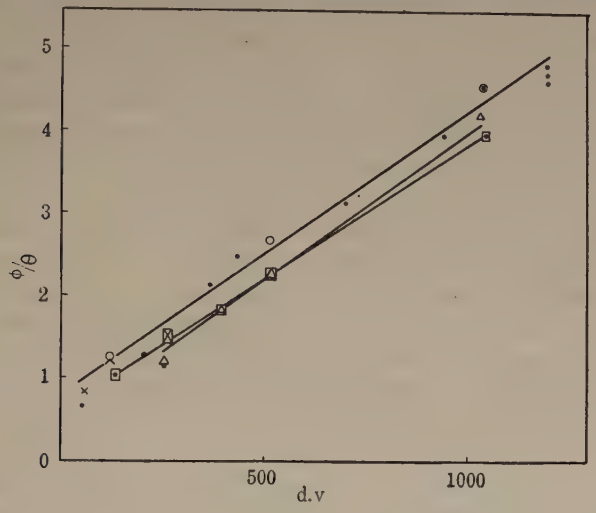

Fig. 9·5.

This is shown by computations [2] of the phase difference produced by the shadow of the head. Second, if the curves passed through the origin, computation shows that the effect would be fundamentally caused by a difference in time of arrival.[3] As will be abundantly emphasized in Chapter X, the phase effect is very important in all binaural acoustic finders, submarine and aerial. Observers are being selected and trained for this work. It is found that train-

[1] Figure 9·5 is taken from a later publication, Stewart, Psychological Monographs, Vol. 31, No. 1; University of Iowa Studies in Psychology, No. VIII. H. M. Halverson, Am. Journ. Psych., **38**, 97 (1927), gives similar results.

[2] See Stewart, Phys. Rev., **15**, 432 (1920), Fig. 4, and accompanying discussion.

[3] If $\varphi = K_1\theta$, then according to the assumed linear relation we have $\varphi \propto f\theta$, wherein f here represents the frequency. Now a phase difference may always be expressed as $2\pi f \cdot \Delta t$, where Δt represents a time interval. A difference of phase, φ, at the ears is thus $2\pi f \cdot \Delta t$, where Δt is the difference in the time of arrival. Substituting $\varphi \propto f \cdot \Delta t$ in $\varphi \propto f\theta$, we have $\Delta t \propto \theta$. That is, θ is independent of frequency and dependent only on the difference in time of arrival of the sound at the ears. Klemm, Arch. f. d. ges. Psych., **40**, 117–146 (1920), emphasizes this difference in time of arrival in the case of impulsive sounds. Such sounds have been made the basis of several psychological studies. Our discussion here is too limited to include them.

ing does lead to improvement. The ability rests not so much on the sensitivity of the ears as upon the alertness of the individual. There is at present a considerable interest in the psychological aspects of both the binaural effects, intensity and phase. It must not be inferred that through the studies of the intensity and the phase effect the phenomenon of binaural localization is thoroughly understood. On the contrary there has been made but a beginning in understanding. That the phase effect cannot be interpreted in terms of intensity is clearly indicated by the fact that in the regions where the intensity effect lapses the phase effect is found to be clearly in evidence, provided the frequency is sufficiently low.[1]

There is one peculiarity of the phase effect which is of interest in practical applications as well as in theory. For most people the binaural phase effect ceases at approximately 1300 cycles. It is found that some experimenters, after considerable experience, can extend this range up to several thousand.[2] The difference of phase effect may be readily observed by using two tuning forks, one at each ear, and listening for the rotation of the phantom sound about the head in front. The forks should beat about once in five seconds to make the effect easily noticeable. If the beats are too rapid, the rotation of the phantom sound is lost and the phenomenon of binaural beats appears. That is, beats similar to monaural beats are heard but with the difference that the sound is not reduced to zero, but only to a minimum. These beats have been studied [3] but their cause is somewhat conjectural. So far as is at present known, a slight deafness does not noticeably impair the usefulness of a binaural phase effect observer.

[1] See Stewart, Phys. Rev., **15**, 432 (1920), and Psych. Mono., l.c.

[2] In certain cases the limitation of the phase effect seems to be only that of audibility. See Halverson, Am. Journ. of Psychology, **38**, 97 (1927). At the higher frequencies there seems to be greater fatigue.

[3] Stewart, Phys. Rev., IX, 6 (1917), pp. 502, 509, 514. C. E. Lane, Phys. Rev., **26**, 401 (1925).

CHAPTER X

Subaqueous Sound Ranging and Signalling

10·1. Sound Signalling in Water.[1]—Advantage of the Acoustic Method.—The importance of an effective method of subaqueous signalling can hardly be overestimated. Four general schemes suggest themselves, viz., optical, magnetic, electrical and acoustic. The effectiveness of each of these will now be considered. The optical method may be dismissed with little discussion, for water is highly opaque to infra-red and ultra-violet and not particularly transparent even for visible light.[2] The magnetic method might appear more possible were it not for the fact that instruments for the detection of very small magnetic fields are not sufficiently rugged for under water use and there is always the added complication of the local magnetic field.

Since sea water is a good conductor of electricity, electromagnetic waves are rapidly absorbed in passing through it, thus greatly diminishing the possibility of an electrical method of signalling. There remains then the acoustic method, and it is found that the sea is relatively a good medium for the propagation of sound waves, a matter discussed in detail in the following section.

10·2. The Sea as an Acoustic Medium.—The advantages and disadvantages of the sea as an acoustic medium will now be discussed. First, with regard to shape it will be noted that the sound waves are confined in a definitely bounded layer with varying cross section, to be sure, but at no place much more than 5 miles thick. This condition in one sense serves to confine the spreading of the wave, though the advantage is somewhat nullified by the fact that the bottom is an absorbing medium and that the surface reflects with change of phase of pressure and displacement (see Sec. 1·17).

[1] This chapter will be devoted primarily to subaqueous sound signalling. The problems peculiar to signalling in air are more appropriately treated in the chapter on atmospheric acoustics. Some of the principles developed in the present chapter, such as binaural localization, etc., will however have application to signalling in air as well as in water.

[2] Thus F. Collin (Le Génie Civil, 79, 375, 1921) quotes W. Bragg's estimate of the visibility distance for visible light in the Mediterranean as of the order of 60 meters; while in the English Channel the corresponding distance is only one-tenth as much.

The latter action is equivalent to an image source of practically equal intensity, but opposite in phase. The source and its image will produce practically zero intensity just within the surface. The closer the source to the surface the greater the region of interference. This suggests at once the desirability of using sources at some distance below the surface and if possible to have a source emitting cylindrical or plane waves. The "beam" transmission has in fact been partially realized by the use of high frequency sources, which will be discussed later in this chapter.

Variations in the homogeneity of the sea water would seem to be of obvious importance. Thus the salt content may change from place to place, and the temperature and the density of the water both change with the depth. This changes the velocity of sound, given, as will be recalled, by

$$c = \sqrt{\frac{1}{\rho K}},$$

where K is the compressibility or reciprocal of the volume elasticity (see (1·30), Sec. 1·13)). An increase in the salt content of 20% brings about a decrease in K of only 5% and therefore an *increase* in c of only about 2.5%. Such an increase in the salt content, however, causes an increase in the density of about 15% and hence a *decrease* in c about *three* times the increase due to the effect of K. At moderate depths the effect of change of salt content on the velocity is therefore rather small. The pressure increases with depth in accordance with the law $p = p_0 + \rho hg$ where p_0 is the pressure at the surface, h the depth, g the acceleration of gravity, and ρ the mean density of the water. It has been found [1] that the compressibility K decreases with increasing pressure, about 1% for 10 atmospheres increase. The increase in density helps to compensate for this in the value $c = \sqrt{1/\rho K}$. Finally the temperature decreases slightly with increasing depth causing an increase in the compressibility [2] of a magnitude of about 2% for a change in temperature of 5° C. in the neighborhood of 0° C. The change in velocity due to temperature is thus very small, particularly when we note how slight the changes in temperature are over a depth of a hundred meters or so.

The net effect of the non-homogeneities in the sea water is thus

[1] See Krummel, Handbuch der Ozeanographie, 1, 237, 298, 1907.

[2] See Bridgman, Proc. Amer. Acad., 48, 310, 1912.

seen to be rather small. As a matter of fact the consistency with which it has been possible to secure measurements on the velocity of sound in sea water is another evidence of the fairly good homogeneity of this medium. Measurements carried out on Nov. 1, 1923 by E. A. Eckhardt[1] off Fort Wright, Long Island Sound, indicate that the only short-period variable affecting the sound velocity is the tidal current. For example, at 2 P.M. with current running he obtained $c = 1494.2$ m/sec (with an error not greater than one part in 25,000), while at 4 P.M. when the current was about zero, he got $c = 1492.3$ m/sec. These represent means of a number of observations in which sound was transmitted over a distance of about 300,000 feet at a depth of 42 feet.

It may be worth mentioning (as indicated above) that in sound signalling near the coast and particularly near river mouths the variations in salt content and the presence of currents are bound to be most noticeable. This is further discussed in the next section in connection with the range.

10·3. The Range.—Theoretical and Experimental Considerations.—In Section 3·2 it was shown that the energy flow in a spherical wave, if there is no other cause of dissipation than the geometrical divergence, decreases inversely as the square of the distance from the source. With cylindrical waves[2] the decrease varies as the inverse *first* power of the distance. A complete treatment of the problem of cylindrical waves is beyond the scope of this book. Nevertheless the law of the decrease in intensity with the distance can be understood by the following relatively simple considerations. We note that the total energy flow per second is equal to the energy flow per second *per unit area* (i.e., the intensity) times the area of the surface through which the flow is taking place. For given energy output the intensity thus varies inversely as the area. But in the case of cylindrical waves the area through which the wave passes is simply $2\pi lr$ where l is the height of the cylinder and r is the distance OP of the surface in question from the axis of the cylindrical source (see Fig. 10·1). The result we have obtained follows at once.

One would expect that both for spherical and cylindrical waves the decrease in intensity would be due mainly to this geometrical spreading. For the decrease due to pure viscosity should be very small. This has already been noted in Sec. 3·3, where it was shown

[1] E. A. Eckhardt, Phys. Rev., 24, 452, 1924.

[2] See Lamb, Hydrodynamics, 1916, p. 520.

that for a plane wave (and the result as far as absorption is concerned will be the same for a spherical or cylindrical wave) the attenuation coefficient, which is the reciprocal of the distance in which the

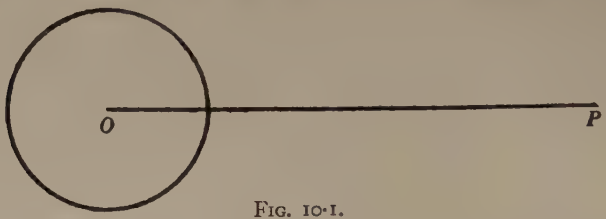

FIG. 10·1.

displacement amplitude is reduced in the ratio $1/e$, is given by

$$\alpha = \frac{2}{3}\frac{\mu k^2}{\rho_0 c},$$

where μ is the coefficient of viscosity of the medium. Now for water at 15° C., $\mu = .0114$ in the usual c.g.s. units. Hence for a wave of 1000 cycles we have

$$\alpha = 10^{-10}$$

approximately. Thus such a wave would travel approximately 100,000 km before being damped in the ratio indicated. For lower frequencies, the distance would be even greater, as the formula indicates. It is clear that the effect of viscosity on the transmission of low frequency waves in water is practically negligible.

It will be of value then to estimate the theoretical value of the range to be expected on the assumption of inverse first or second power falling off.

Let us suppose we have a source with an output of 100 watts. Then if the propagation is in a spherical wave the intensity at distance r from the source will be given by

$$\frac{100e^{-\alpha r}}{4\pi r^2}$$

in watts per cm^2. At a distance of 1000 km this would become practically 8×10^{-16} watts/cm^2. Hence assuming a receiving instrument of this sensitivity, the figure of 1000 km would represent the theoretical range.

Consider further the case of a cylindrical source having an output of 100 watts for each 20 meters of its height. The intensity at

a distance r from the source will be

$$\frac{100e^{-\alpha r}}{(2\pi)2000r}.$$

Using the same value of $\alpha = 10^{-10}$, it is seen that if 10^{-15} watts/cm^2 represents the minimum detectable intensity, this limit will be reached for $r = 5 \times 10^{10}$ cm roughly or 500,000 km.

As a matter of fact the actual experimentally observed values of the range fall far below either of the two theoretical values computed above. Thus under the best conditions ranges have not been reported greater than about 250 km.[1]

It is clear that the falling off in intensity in sound waves in water is not due to spherical or cylindrical divergence. On the contrary all the evidence points to exponential damping or absorption. This, however, can not be due merely to viscosity. Hence the reason for the shortness of the ranges commonly observed must be due to non-homogeneities in the water. This view is confirmed by the observation that, other things being equal, the range is invariably greater in winter than in summer, the ratio being approximately three to one. This is exactly what would be expected from the behavior of the temperature gradient. Thus in the summer the surface water is warmer than the lower depths causing a bending of the waves downward. In very deep water this would clearly lower the range. In shallow water the same result would occur since the sea bottom is a poor reflector and would absorb a large amount of the incident sound. During the colder months, on the other hand, the temperature gradient is much less marked or may actually be reversed, leading to a longer range, since the wave would be bent upward and the reflection from the water surface is almost perfect.

The effect of variation in the salt content on the range is very slight on the open sea but may prove very noticeable near the mouth of a large river. Even more pronounced is the influence of eddies and currents which can cut off completely sound signals directed across them. Thus Barkhausen and Lichte (loc. cit.) report instances in which ranges of but two or three kilometers were attained using apparatus with which 50–100 kilometers had previously been easily obtained. They note that, in general, ranges are

[1] See Aigner, Unterwasserschalltechnik, Berlin, 1922, p. 52. Also see Barkhausen und Lichte, Ann. der Physik, 62, 485, 1920.

less near the sea coast, as might be expected from the preceding discussion.

The effect of change in density with depth and of compressibility is noted only in deep sea signalling and is not very marked (see Aigner, p. 50).

The fact that the range is controlled by exponential decrease in intensity has the practical result, as pointed out by Barkhausen and Lichte (loc. cit.), that the increase in range to be expected by increasing the power supplied by the transmitter is negligible compared with what could be obtained if the intensity fell off with the inverse first or second power of the distance. This, of course, applies to what may be called low-frequency signalling (i.e., up to 1000 cycles). High-frequency signalling will be discussed later.

10·4. Transmission from and to Water. Application of Stethoscope Principle.—It has already been pointed out in Section 4·4 that in the ordinary passage of sound from water to air or vice versa the amount transmitted is only about 0.12%, an almost negligible quantity. It was pointed out in the same section, however, that by employing a suitably constructed stethoscope with a very thin air chamber the power transmission becomes

$$P_r = \frac{4m_1r_{12}}{(m_1 + r_{12})^2}, \qquad (10\cdot1)$$

in which $m_1 = S_2/S_1$, the ratio of the cross section of the stethoscope tube to that of the air chamber, and $r_{12} = R_2/R_1$ = ratio of specific acoustic resistances of air and water. This quantity equals 3×10^{-4}. We thus get unit transmission for

$$S_2/S_1 = 3 \times 10^{-4}.$$

Though this would be in practice a somewhat exaggerated case, the principle is of obvious application.

In Section 4·6 was discussed the use of an intervening medium between the water and the air chamber of the stethoscope, and it was shown that if this medium has a specific acoustic resistance which is a geometric mean between the values for air and water respectively the transmission will be unity if the thickness of the medium is one-quarter of the wave length (in this medium). It was pointed out that this thickness condition imposes on the transmission a selectivity which is very pronounced. This is probably an important item in the explanation of the observed sensitivity [1] of

[1] See H. A. Wilson, Phys. Rev., 15, 178, 1920.

sound receivers with thick rubber nipples (see Sec. 10·11 for further details).

When sound travels from one medium to another through an intervening medium (e.g., from water to air via rubber or iron) the power transmission is independent of the properties of the intervening medium provided the thickness of the medium is an integral multiple of one-half wave length (in the medium itself). One might therefore suppose that for a very thin intervening medium this would still be true (for $\sin kl = 0$ under these circumstances as well). But, as was emphasized in the concluding paragraph of Section 4·3, a very thin sheet of a substance no longer acts as a true *medium* and the analysis from which the above result was obtained no longer applies. This matter is very important in connection with the transmission of sound through the hull of a ship. The latter does not act like a medium but vibrates as a whole. It is, however, a well-known experimental fact that the vibration of the hull is of the same order of magnitude as that of the water. In fact we can see by very simple considerations that for the resonance frequency of the hull the transmission from the water to the air inside takes place more or less independently of the existence of the hull. Thus if the vibrating area of the hull is S and the *effective* mass is M with a stiffness coefficient F, and we denote its displacement by ξ, then, neglecting damping the expression for the pressure on the area S of the hull is given by

$$pS = M\ddot{\xi} + F\xi. \tag{10·2}$$

Now let the incident and reflected plane wave displacements in the water at the hull be

$$A_1 e^{i\omega t} \quad \text{and} \quad B_1 e^{i\omega t},$$

while the transmitted wave displacement in the air on the other side of the hull is

$$A_2 e^{i\omega t}.$$

Now the boundary condition of pressure continuity demands that the excess pressure in the incident and reflected waves plus the pressure due to the vibration of the hull shall equal the pressure of the transmitted wave in the air. But this condition is

$$iR_1\omega(A_1 + B_1) + \frac{M}{S}\ddot{\xi} + \frac{F}{S}\xi = \omega i R_2 A_2, \tag{10·3}$$

where R_1 and R_2 are the specific acoustic resistances of the water and air respectively. The continuity of displacement demands

$$A_1 - B_1 = A_2.$$

The solution of the above equations for A_2 in terms of A_1 would in general require a knowledge of ξ for the particular frequency being transmitted. But for the transmission of the frequency for which the hull (or the section of it being considered) is in resonance, we have at once

$$M\ddot{\xi} + F\xi = 0, \tag{10·4}$$

whence the boundary equations become:

$$R_1(A_1 + B_1) = R_2A_2,$$
$$A_1 - B_1 = A_2,$$

leading to

$$A_2 = \frac{2A_1}{r_{12} + 1}, \tag{10·5}$$

where $r_{12} = R_2/R_1$, the relative specific acoustic resistance. The power transmission

$$P_r = r_{12}\frac{|A_2|^2}{|A_1|^2} = \frac{4r_{12}}{(r_{12} + 1)^2} \tag{10·6}$$

is thus the same as if there were *no* intervening hull.

A discussion of the vibration of plates will be given in Section 10·6. It will be of interest to note here that the lowest natural frequency in air of a metal plate of thickness h and radius a in which the velocity of sound is c is given by [1]

$$\nu = 0.47\frac{hc}{a^2}, \tag{10·7}$$

while the frequency of the same plate *in water* becomes

$$\nu' = \frac{0.47}{\sqrt{1 + \beta}}\frac{hc}{a^2}, \tag{10·8}$$

where

$$\beta = .67\frac{\rho}{\rho_1}\frac{a}{h}, \tag{10·9}$$

where ρ is the density of water and ρ_1 the density of the material composing the plate. To take a special case, suppose the plate is

[1] See Lamb, Proc. Roy. Soc., 98, 205, 1920.

$\frac{1}{8}$ inch thick and 7 inches in diameter and is made of iron, whence $c = 5.23 \times 10^5$ cm/sec and $\rho_1 = 7.8$. The value of the frequency in air is thus 1013 cycles while for water it is reduced to 550 cycles. If the hull were one inch thick the diameter of an area which would vibrate in water with the same natural frequency is given by solving for a in terms of h in the expression for ν', where β is replaced by its value in terms of a and h. An equation of the fifth degree results. For the case at hand its approximate solution yields $a = 28.5$ cm for the radius if $h = 2.54$ cm and the same under water frequency is desired (namely 550 cycles). Naturally for larger thicknesses the resonating area must be materially increased. Moreover the resonance is fairly sharp.[1] Nevertheless for the case of the ship's hull the general conclusion may be reached that it will not interfere markedly with the transmission from water to air unless it is very thick. Hence the use of the hull for the attachment of receiving instruments or even as a source of sound is practical.

In a discussion of the transmission from water to air it is not inappropriate to call attention to the use of "light bodies." It will be recalled (see Sec. 4·1) that when sound passes from an acoustically dense medium to an acoustically rare medium (e.g., water to air) the transmitted amplitude is approximately twice the incident amplitude, though of course very little energy is transmitted. It is therefore clear that if a body much lighter than water were submerged in water (by the use of suitable constraints) sound waves incident from the water would cause the body to vibrate with a larger vibration-amplitude than that of the incident wave provided that the constraints employed to keep the body submerged do not act to retard materially its motion. We can look at the matter most simply in this way. The sound waves exert approximately the same total force on the submerged body that they would on an equal volume of water of the same shape assuming that the dimensions of the body are much less than the wave length so that the volume of water may be conceived approximately to move as a whole. Let M_b be the mass of the body and M_w be the mass of the displaced water. Let the *extra mass* of the body due to the fact that it is vibrating in water (see Sec. 10·5 for a more detailed discussion of this quantity) be M_e. This will depend solely on the size and shape of the body and the direction in which it vibrates with reference to its shape (e.g., if an ellipsoid whether it vibrates in a direction paral-

[1] See Lamb, loc. cit., p. 210.

lel to its major axis or perpendicular thereto). Let the maximum acceleration of displacement for the case where the water alone is concerned (i.e., the displacement acceleration for the sound wave) be a_w, while that for the light body is a_b. The fact, stated above, of the approximate equality of the force then leads at once to the equation

$$(M_b + M_e)a_b = (M_w + M_e)a_w. \qquad (10{\cdot}10)$$

But we have the relations:

$$a_b = -\omega^2 A_b,$$
$$a_w = -\omega^2 A_w,$$

where A_b and A_w are the displacement amplitudes of the body and the incident wave respectively, and ω is $2\pi \times$ frequency, as usual. Hence for the amplitude ratio we have

$$\frac{A_b}{A_w} = \frac{M_w + M_e}{M_b + M_e}, \qquad (10{\cdot}11)$$

which is, of course, greater than unity if $M_b < M_w$. As pointed out above M_e must be computed from the size and shape and orientation of the body. For example if the body is ellipsoidal, M_e is much greater when the major axis is perpendicular to the direction of the sound wave than when the major axis lies in the direction of the wave. In some practical experiments conducted by Wood and Young [1] this was shown to result in a directional effect such that the ratio A_b/A_w for the "head on" direction is greater than that for the "broadside" direction in the ratio of approximately 5 to 2. This seems about the maximum directional effect to be obtained with the use of ellipsoidal bodies. Since the directional effect of single microphones may be perhaps ten times as much, the practical value of the use of the ellipsoid is very small. The "light body" principle in general, however, has been of value in hydrophone research.

10·5. Sources of Sound. Radiation from a Sphere.—Before taking up practical sources of sound for subaqueous signalling it will be desirable to discuss the influence of the medium on the source. In water this effect may be considerable.

A rather simple illustration is provided by the radial vibration of a sphere, emitting spherical waves into the surrounding medium.

[1] A. B. Wood and F. B. Young, Proc. Roy. Soc., 100, 252, 1921–22.

The velocity potential at distance r from a point source is (Sec. 3·2)

$$\varphi = A/r \cdot e^{i(\omega t - kr)}.$$

Now the corresponding quantity for the case of a spherical source of radius a is

$$\varphi = A/r \cdot e^{i[\omega t - k(r-a)]}, \tag{10·12}$$

where r is now the distance from the center of the sphere. For direct substitution shows that this satisfies the general wave equation (1·16) and the boundary condition that for $r = a$ the particle velocity will be the radial velocity of the spherical surface itself, namely,

$$\dot{\xi}_a = \dot{\xi}_0 e^{i\omega t}.$$

Thus

$$\dot{\xi}_r = -(1/r + ik)A/r \cdot e^{i[\omega t - k(r-a)]}, \tag{10·13}$$

whence the boundary condition is satisfied if we choose A so that

$$A = -\frac{\dot{\xi}_0 a^2(1 - ika)}{1 + k^2a^2}. \tag{10·14}$$

Substitution yields

$$\varphi = -\frac{\dot{\xi}_0 a^2(1 - ika)}{1 + k^2a^2} \cdot \frac{1}{r} \cdot e^{i[\omega t - k(r-a)]}. \tag{10·15}$$

The excess pressure at the surface of the sphere due to the radiation is (see again Sec. 3·2)

$$p_a = -\rho_0 \dot{\varphi}_a = \frac{(i + ka)\omega\rho_0 a}{1 + k^2a^2} \dot{\xi}_0 e^{i\omega t}. \tag{10·16}$$

The specific acoustic impedance at the surface of the sphere then becomes by definition

$$Z_S = \frac{p_a}{\dot{\xi}_0 e^{i\omega t}} = \frac{(ka + i)\omega\rho_0 a}{1 + k^2a^2}, \tag{10·17}$$

which yields the specific acoustic resistance

$$Z_{S1} = \frac{\omega\rho_0 ka^2}{1 + k^2a^2} \tag{10·18}$$

and the specific acoustic reactance

$$Z_{S2} = \frac{\omega\rho_0 a}{1 + k^2a^2}, \tag{10·19}$$

which is in this case an *inertance*, since it is positive. It must be

emphasized that Z_S is a *radiation* impedance due entirely to the fact that the sphere is vibrating in the medium. The *extra* force (or force of reaction) on the sphere may then be written

$$F_a = Z_{S1} \cdot S\dot{\xi}_a + iZ_{S2} \cdot S\dot{\xi}_a = Z_{S1} S\dot{\xi}_a + \frac{Z_{S2} S}{\omega} \ddot{\xi}_a, \quad (10{\cdot}20)$$

where S is the surface area of the sphere and we utilize the fact that $\ddot{\xi}_a = i\omega\dot{\xi}_a$. The coefficient of $\dot{\xi}_a$ in the above equation is the radiation resistance coefficient while the coefficient of $\ddot{\xi}_a$ is the radiation inertia coefficient. Thus the equation of radial motion of the sphere should now be written

$$\left(M + \frac{Z_{S2} S}{\omega} \right) \ddot{\xi}_a + (R + Z_{S1} S)\dot{\xi}_a + f\xi_a = Q \quad (10{\cdot}21)$$

if M, R and f are the equivalent *mechanical* inertia, resistance and stiffness coefficients of the sphere respectively, and Q is the driving force. Written out in full the radiation inertia and resistance coefficients are

$$\frac{SZ_{S2}}{\omega} = \frac{4\pi\rho_0 a^3}{1 + k^2 a^2} \quad (10{\cdot}22)$$

and

$$SZ_{S1} = \frac{4\pi a^4 \omega \rho_0 k}{1 + k^2 a^2} . \quad (10{\cdot}23)$$

We shall consider these quantities for the two special cases of *low* and *high* frequencies respectively. In the former case (if the source is reasonably small) $k^2 a^2$ will be small compared with unity and we have

$$\frac{SZ_{S2}}{\omega} = 4\pi a^3 \rho_0, \quad (10{\cdot}24)$$

while

$$SZ_{S1} = \frac{4\pi a^4 \omega^2 \rho_0}{c} . \quad (10{\cdot}25)$$

The radiation inertia coefficient for low frequencies may thus be of considerable magnitude, representing, as it does, the mass of a volume of the medium three times the volume of the vibrating sphere. For $\rho < 3\rho_0$, where ρ is the mean density of the sphere (which of course may be a spherical shell) the radiation inertia will actually be larger than the ordinary mass of the vibrator. On the other hand, for low frequencies SZ_{S1} will be relatively small.

For very high frequencies k^2a^2 will be large compared with unity and the radiation inertia becomes

$$\frac{4\pi\rho_0 a}{k^2} = \frac{4\pi\rho_0 a c^2}{\omega^2}, \tag{10·26}$$

which is negligible compared with the radiation inertia in the low frequency case. The radiation resistance now is

$$4\pi a^2 \rho_0 c, \tag{10·27}$$

which is independent of frequency and relatively large. In fact we note that the specific resistance in this case is simply

$$Z_{S1} = \rho_0 c, \tag{10·28}$$

which is, however, the specific acoustic resistance for a plane wave.

The practical result, as will now be shown, is that the vibrating sphere is more efficient as a generator of sound radiation at high frequency than at low. At low frequencies the principal effect of the medium is to increase the effective inertia of the sphere and thereby increase its kinetic energy of vibration for a given fixed value of $\dot{\xi}_a$ without increasing the energy dissipated. At high frequencies, on the other hand, the effect is to increase the dissipation term in the equation of motion and thus allow more radiation. In fact we recall from Section 2·1 (eq. (2·10)) that the average rate of dissipation of energy by any oscillator is given by

$$R'\overline{\dot{\xi}^2},$$

where R' is the total resistance coefficient. In the present case, the corresponding quantity is

$$\left(R + \frac{4\pi a^4 k^2 c \rho_0}{1 + k^2 a^2}\right) \overline{\dot{\xi}_a^2}.$$

It is at once seen that this increases with the frequency if the amplitude of $\dot{\xi}_a$ remains fixed. Considering the flow of energy in a plane wave (see Sec. 1·15) it will be noted that the flow of energy is proportional to the square of the amplitude of the particle velocity and is otherwise independent of the frequency. Thus the condition here of the constant amplitude of $\dot{\xi}_a$ shows the intrinsic dependence of radiation upon frequency in the present case. We have here one indication of the possible efficiency of high frequency signalling in water.

10·6. Sources of Sound. Vibrating Plate.—The problem of the vibration of a diaphragm or plate in water is most easily handled by considering the plate replaced by a piston [1] all points of which have the same displacement at any instant. Consider the piston as of circular cross section with radius a, the center being at O. In order to find the influence of the radiation into the surrounding medium on the vibration of the plate, we must calculate the velocity potential at any point P', distant r from the center. From this the excess pressure can be calculated and hence the total radiation reaction force on the whole plate. In the calculation of the velocity potential we can consider each point of the plate as a point source of spherical waves emitted on one side only. The contribution to the velocity potential at P by a surface element dS at P' distant r_1 from P is given by

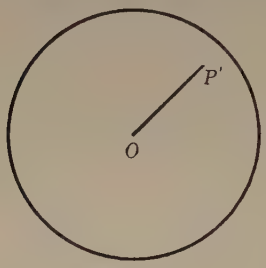

FIG. 10·2.

$$\frac{AdS}{r_1} e^{i(\omega t - kr_1)},$$

wherein A/r_1 is the amplitude produced per unit area; whence the total velocity potential at P is

$$\varphi_P = Ae^{i\omega t} \int\int \frac{e^{-ikr_1}}{r_1} dS. \qquad (10\cdot29)$$

The constant A may be evaluated rather simply in this special case by the condition that the displacement velocity of the plate (uniform over the whole surface) is $\dot{\xi}_0 e^{i\omega t}$. Consider the little hemisphere about P' of radius r_0 and take this as a hemispherical source of sound. The velocity potential due to this source at a point distant z from the point P' will then be

$$\frac{2\pi r_0^2 A}{z} e^{i[\omega t - k(z-r_0)]},$$

wherein the dS has been replaced by $2\pi r_0^2$, the area of the assumed hemisphere. Now the resulting velocity at the surface of the hemisphere will be

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z=r_0} = \dot{\xi}_{r_0} = -2\pi r_0^2 \left(\frac{1}{r_0} + ik\right) \frac{A}{r_0} e^{i\omega t}. \qquad (10\cdot30)$$

[1] See, for example, Crandall, Vibrating Systems and Sound, 1926.

By the above condition we have

$$\lim_{r_0 \doteq 0} \dot{\xi}_{r_0} = \dot{\xi}_0 e^{i\omega t}.$$

Therefore,

$$\dot{\xi}_0 = \lim_{r_0 \doteq 0} \left[-2\pi r_0^2 \left(\frac{1}{r_0} + ik \right) \frac{A}{r_0} \right] = -2\pi A, \qquad (10{\cdot}31)$$

whence

$$A = -\dot{\xi}_0 / 2\pi. \qquad (10{\cdot}32)$$

It may be noted that Rayleigh [1] has proved the general theorem that the velocity potential due to any vibrating surface radiating into the region on one side only at a point is given by

$$\varphi = -\frac{1}{2\pi} \int\int \frac{\partial \varphi}{\partial n} \frac{e^{-ikr}}{r} dS, \qquad (10{\cdot}33)$$

where r denotes the distance from the element dS to the point in question and $\partial\varphi/\partial n$ is the maximum displacement velocity normal to the surface at dS. Our expression (10·29) is a special case of the above in which $\partial\varphi/\partial n = \dot{\xi}_0 e^{i\omega t}$ and is constant.

The actual calculation of the reaction force F from φ_P involves another surface integration so that the whole evaluation necessitates a double surface integration. This will not be carried out here. The calculation may be found in Rayleigh [2] and we shall confine ourselves to quoting and discussing the result, which is

$$F = \pi a^2 \rho_0 c \dot{\xi}_0 e^{i\omega t} \cdot \left[1 - \frac{J_1(2ka)}{ka} \right] + \frac{i\omega\pi\rho_0 \dot{\xi}_0}{2k^3} e^{i\omega t} \cdot K_1(2ka), \qquad (10{\cdot}34)$$

wherein $J_1(2ka)$ is the Bessel's function of the first order and argument $2ka$, while

$$K_1(2ka) = \frac{2}{\pi} \left[\frac{(2ka)^3}{3} - \frac{(2ka)^5}{3^2 \cdot 5} + \frac{(2ka)^7}{3^2 \cdot 5^2 \cdot 7} \cdots \right].$$

By definition the specific acoustic impedance then is

$$Z_S = \rho_0 c \left[1 - \frac{J_1(2ka)}{ka} \right] + \frac{i\omega\rho_0}{2a^2k^3} K_1(2ka), \qquad (10{\cdot}35)$$

wherein the real part is Z_{S1}, the specific acoustic resistance, while the

[1] Rayleigh, Sound, Vol. II, 1916, § 278.

[2] Rayleigh, Sound, Vol. II, 1916, § 302. See also Crandall, p. 143 ff, for a rather simple derivation.

imaginary part is Z_{S2}, the specific acoustic reactance or inertance. We proceed to discuss these for high and low frequencies respectively.

$J_1(2ka)$ gradually decreases as ka increases.[1] Hence for high frequencies, the resistance becomes approximately

$$Z_{S1} = \rho_0 c,$$

the same as for a plane wave, a result already obtained in Sec. 10·5 for the case of a vibrating sphere.

For large values of $2ka$, Rayleigh [2] shows that

$$K_1(2ka) \doteq \frac{2}{\pi}(2ka).$$

Hence for high frequencies we have

$$Z_{S2} = \frac{2\omega\rho_0}{\pi a k^2} \tag{10·36}$$

corresponding to a radiation inertia (see Sec. 10·5 eq. (10·24))

$$\frac{SZ_{S2}}{\omega} = \frac{2\rho_0 a}{k^2} = \frac{2\rho_0 a c^2}{\omega^2}. \tag{10·37}$$

This is, of course, small for a not too large. It is instructive to compare this value with the corresponding one for the sphere, namely $4\pi\rho_0 a/k^2$, the ratio of the former to the latter being $\frac{1}{2}\pi$.

It is the low frequency case which is of most interest to us at this point. For small values of ka, there results

$$1 - \frac{J_1(2ka)}{ka} = \frac{k^2a^2}{2}$$

for

$$J_1(2ka) = \frac{2ka}{2}\left[1 - \frac{(2ka)^2}{2\cdot 2^2} + \cdots\right],$$

whence we get

$$Z_{S1} = \tfrac{1}{2}\rho_0 c k^2 a^2 \tag{10·38}$$

and the radiation resistance coefficient is

$$SZ_{S1} = \tfrac{1}{2}\pi\rho_0 c k^2 a^4. \tag{10·39}$$

This should be compared with the corresponding quantity, $4\pi a^4 k^2 \rho_0 c$ (10·25) for the sphere. The ratio of the former to the latter is $\frac{1}{8}$. It is of interest to note that if we consider a vibrating *hemisphere* we

[1] See, for example, Rayleigh, loc. cit.

[2] Rayleigh, Sound, Vol. II, 1916, loc. cit.

should have as radiation resistance coefficient $2\pi a^4 k^2 \rho_0 c$. The ratio is then reduced to $\frac{1}{4}$ and we can say that the radiation resistance of the plate is *equal* to that of a vibrating hemisphere of the same area.

For ka small we see that $K_1(2ka)$ reduces to the first term in the expansion above and hence the radiation inertia coefficient becomes

$$\frac{S \cdot Z_{S2}}{\omega} = \frac{8}{3} \rho_0 a^3. \qquad (10{\cdot}40)$$

This is thus $2/\pi$ times the mass of a sphere of the medium of the same radius as the plate.

If the purely *mechanical* resistance coefficient or damping factor is small compared with the *radiation* resistance coefficient, and ka is small, the equation of motion of the plate with effective mechanical mass m and stiffness coefficient f is

$$(m + \frac{8}{3} \rho_0 a^3)\ddot{\xi} + \tfrac{1}{2}\pi \rho_0 c k^2 a^4 \dot{\xi} + f\xi = Q, \qquad (10{\cdot}41)$$

where Q is the driving force. The average rate of radiation of sound energy into the water is

$$\tfrac{1}{4}\pi \rho_0 c k^2 a^4 \left| \dot{\xi}_{\max} \right|^2 \qquad (10{\cdot}41a)$$

and this increases with the frequency, bearing out the remarks made at the close of the previous section.

It will be of interest to note the effect on the frequency of the radiation inertia at low frequencies. The natural frequency is given approximately by

$$\nu = \frac{1}{2\pi} \sqrt{\frac{f}{m + \frac{8}{3} \rho_0 a^3}}. \qquad (10{\cdot}42)$$

We see that if the radiation inertia is equal to the effective mechanical mass, the frequency in water is only $1/\sqrt{2}$ or .707 times the frequency in air. In order that the frequency be not lowered by more than 10% the radiation inertia coefficient must not be more than 20% of the effective mechanical mass.

The effect of the radiation resistance on the frequency will be measured by the ratio

$$\frac{\pi^2 \rho_0^2 c^2 k^4 a^8}{8m^2} = \frac{\pi^2 \rho_0^2 \omega^4 a^8}{8m^2 c^2},$$

since the more exact expression for the natural frequency is (see Sec. 2·2)

$$\nu = \frac{1}{2\pi}\sqrt{\frac{f}{m + \frac{8}{3}\rho_0 a^3} - \frac{\pi^2 \rho_0^2 c^2 k^4 a^8}{16 M^2}}, \qquad (10\cdot43)$$

where $M = m + 8/3 \cdot \rho_0 a^3$. Let us take a special case in which $a = 10$ cm, $\omega = 1000$. The extra mass is then 2667 gms and the second term above becomes (using $c = 14.5 \times 10^4$ cm/sec and $\rho_0 = 1$ gm/cm^3),

$$\frac{10^{10}}{3{\cdot}4m^2}.$$

If $m = 10^3$ gm we get this to equal 2.2×10^2. But this will still be only a small correction to the term

$$\frac{f}{m + \frac{8}{3}\rho_0 a^3}$$

in this case.

To apply the preceding results to an actual vibrating diaphragm, which, if it is clamped at the edges, does not vibrate as a piston, we must ascertain the area and effective mass of the equivalent piston. For a given driving force the rate of energy radiation (or radiation damping) can then be calculated.[1]

The effect on the motion of a single circular diaphragm radiating into a semi-infinite medium produced by other diaphragms in the vicinity has been studied theoretically by Wolff and Malter [2] who find that due to differences in phase the diaphragms react on each other to increase the efficiency of radiation at low frequencies.

10·7. Practical Low Frequency Sources.—Several types of instruments have been devised for generating sound under water. Of the purely mechanical variety we may mention the submarine siren built on the same principle as the ordinary air siren only with jets of water passing through holes in a vibrating plate or series of plates. The siren has proved to be inferior to the electromagnetic transmitter and is now rarely used.

The electromagnetic sound generator may be one of two types, viz., the *electromagnet* or telephone receiver type and the *moving coil* or galvanometer type. In the first, alternating current is supplied

[1] See Crandall, Vibrating Systems and Sound, p. 36.

[2] Phys. Rev., 33, 1061, 1929.

to the winding of a large electromagnet arranged so as to attract a heavy diaphragm clamped along the periphery.[1] In the second alternating current is supplied to a coil surrounded by a magnetic field and directly connected to the diaphragm. A modification of this, known as the Fessenden oscillator, is probably the most efficient and generally satisfactory of all underwater low frequency sound generators. It will be briefly described here. The principle of the device can be understood by reference to the diagram (Fig. 10·3)[2] which shows a cross section of the oscillator. The outer part consists of a bipolar magnet energized by direct current through field coils shown in the figure by double cross hatching. The pole pieces are rigidly connected to the steel diaphragm which is $\frac{5}{8}''$ thick and shown at the right of the figure. In the gap is an iron core surrounded by a coil of wire wound in one direction about one half of the length of the core and in the reverse direction about the other half. Alternating current of the frequency desired (generally 540 cycles) is supplied to this coil. In the narrow space between the coil and the magnetic pole pieces is a *copper* cylinder fastened at each end to discs which in turn are secured to a shaft in the middle along the axis of the cylinder. The front disc (at the right in the diagram) is rigidly attached to the diaphragm. The action is as follows. Alternating current traversing the central coil sets up eddy currents in the copper cylinder moving more or less in circles about the cylinder in planes perpendicular to its length. These currents cut the lines of force of the magnet and are opposite in direction over one half of the cylinder from what they are over the other. The result is that during one alternation of the current the cylinder is pushed forward while during the following alternation it is pushed

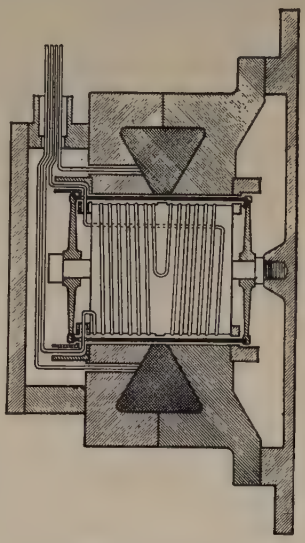

FIG. 10·3. Diagrammatic Cross-section of Oscillator

[1] For a more complete description of such an instrument, see Drysdale et al., Mechanical Properties of Fluids, 1924, p. 304. See also Aigner, loc. cit., p. 179.

[2] Taken with kind permission from a publication of the Submarine Signal Company of Boston, Mass., 1917.

backward, thus causing the diaphragm to vibrate with the same frequency as the current. The device has the additional advantage that the central coil winding is practically non-inductive whence a high power-factor results with corresponding gain in efficiency. An acoustic radiation of 500 watts is available with this instrument under good working conditions at an efficiency of about 50%. The following figure (Fig. 10·4)[1] indicates the resonance characteristics of the device both in air and water.

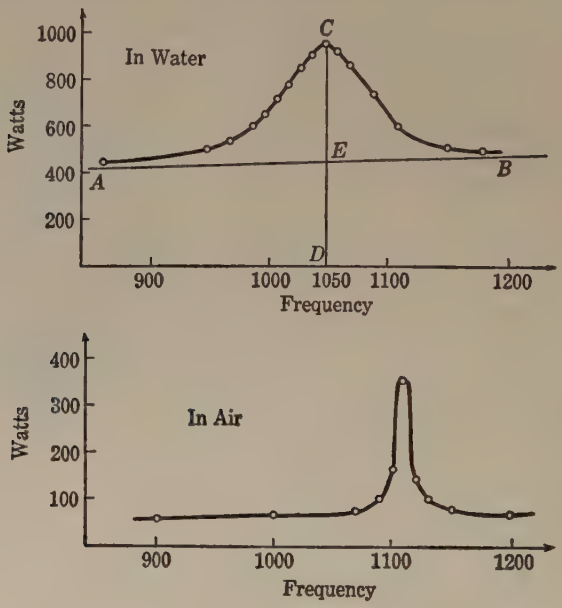

Fig. 10·4.

10·8. High Frequency Sound Radiation.—We now discuss the radiation of sound into a medium by a diaphragm or plate vibrating at high frequency, i.e., such that the wave length of the resulting radiation is small compared with the dimensions of the plate. We should expect from the discussion in Sections 1·2 and 1·3 (Chap. I) that the smaller the wave length relative to the diameter, the more nearly will the resulting sound be confined to a "beam" with little sidewise spreading. This in addition to the greater radiation efficiency of the plate at high frequencies already emphasized in

[1] Taken with kind permission from F. Aigner, Unterwasserschalltechnik, Berlin, Verlag von M. Krayn, 1922.

Section 10·6 constitutes the principal advantage of the modern high-frequency signalling or supersonics.

To consider this matter in a little greater detail, i.e., to discover how the divergence of the "beam" depends on the relative magnitudes of plate and wave length, we take again the vibrating plate of Section 10·6 now represented in perspective in Fig. 10·5. Let the

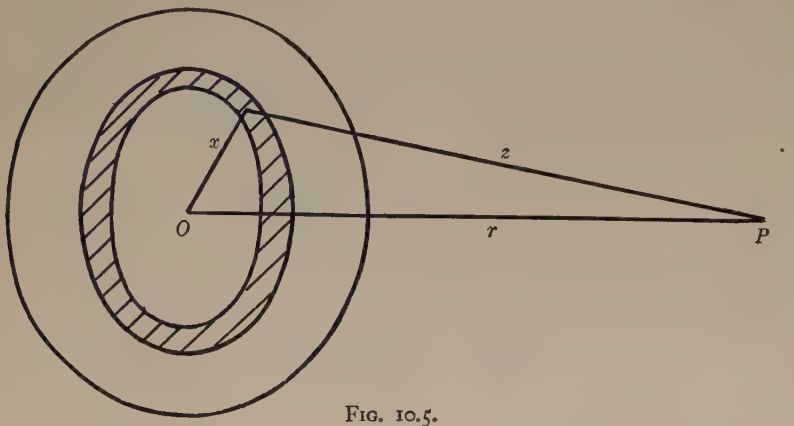

FIG. 10.5.

radius of the equivalent piston by which the plate has been replaced be a. Let it be desired to find the velocity potential at point P on the axis of the piston distant r from the center. This will be the sum of contributions from all area elements of the disc, such as the shaded ring. Thus we can apply the Rayleigh formula (eq. (10·33) Sec. 10·6)

$$\varphi_P = -\frac{1}{2\pi}\int\int \frac{\partial \varphi}{\partial n}\frac{e^{-ikr}}{r}\,dS. \qquad (10\cdot44)$$

Since in the present case

$$\frac{\partial \varphi}{\partial n} = \dot{\xi}_0 e^{i\omega t}$$

(a constant over the whole surface of the disc by the piston assumption), $dS = 2\pi x dx$ and z is the variable, we have

$$\varphi_P = -\frac{\dot{\xi}_0 e^{i\omega t}}{2\pi}\int_0^a \frac{e^{-ikz}}{z}\cdot 2\pi x dx. \qquad (10\cdot45)$$

Introducing $x = \sqrt{z^2 - r^2}$, this becomes

$$\varphi_P = -\dot{\xi}_0 e^{i\omega t}\int_r^{\sqrt{a^2+r^2}} e^{-ikz}dz = -\dot{\xi}_0 e^{i\omega t}\frac{i}{k}\left[e^{-ik\sqrt{a^2+r^2}} - e^{-ikr}\right]. \qquad (10\cdot46)$$

Now let $a = m\lambda$ where m is some number greater than unity. Now the pressure at P is

$$p_r = -\rho_0\dot{\varphi}_P = -\rho_0 c\dot{\xi}_0 e^{i\omega t}\left[e^{-ik\sqrt{r^2+m^2\lambda^2}} - e^{-ikr}\right]. \quad (10{\cdot}47)$$

The power transmission per unit area at P, that is, the *intensity* is (see Sec. 1·15)

$$I = \frac{\overline{p_r^2}}{\rho_0 c},$$

where the real part of p_r is of course meant, and the bar indicates average over time. We have for the real part of p_r

$$p_r = -\rho_0 c\dot{\xi}_0\left[\cos k\sqrt{r^2+m^2\lambda^2} - \cos kr\right]\cos\omega t - \rho_0 c\dot{\xi}_0\left[\sin k\sqrt{r^2+m^2\lambda^2} - \sin kr\right]\sin\omega t.$$

Introducing the identities

$$\cos k\sqrt{r^2+m^2\lambda^2} - \cos kr = -2\sin\frac{k}{2}\left(r+\sqrt{r^2+m^2\lambda^2}\right)\cdot\sin\frac{k}{2}\left(\sqrt{r^2+m^2\lambda^2}-r\right)$$

and

$$\sin k\sqrt{r^2+m^2\lambda^2} - \sin kr = 2\cos\frac{k}{2}\left(r+\sqrt{r^2+m^2\lambda^2}\right)\cdot\sin\frac{k}{2}\left(\sqrt{r^2+m^2\lambda^2}-r\right),$$

squaring and averaging we find

$$I = 2\rho_0 c\dot{\xi}_0^2\sin^2\frac{k}{2}\left(\sqrt{r^2+m^2\lambda^2}-r\right). \quad (10{\cdot}48)$$

If r is large compared with $m\lambda$, we can approximate by putting

$$\sqrt{r^2+m^2\lambda^2} = r\left(1+\frac{m^2\lambda^2}{2r^2}\right).$$

Then we have for I

$$I = 2\rho_0 c\dot{\xi}_0^2\sin^2\frac{\pi m^2\lambda}{2r}, \quad (10{\cdot}49)$$

recalling that $k = \omega/c = 2\pi/\lambda$. It is seen that for values of r greater than $m\lambda$ but less than or of the same order of magnitude as $m^2\lambda$, the intensity may be large. Strictly speaking in this range the intensity runs through a series of maxima and minima, and does not fall off continuously with increasing r. For values of $r > m^2\lambda$, on

the other hand, the intensity on the axis falls off rapidly without oscillations. In fact we have then for large r

$$I = 2\rho_0 c \dot{\xi}_0^2 \cdot \frac{\pi^2 m^4 \lambda^2}{4r^2} \qquad (10{\cdot}50)$$

approximately, and the intensity thus falls off inversely as the square of the distance [1] showing that the radiation is now behaving as a spherical wave. A rough picture of what takes place is afforded by the sketch in Fig. 10·6. Here O represents the center of the

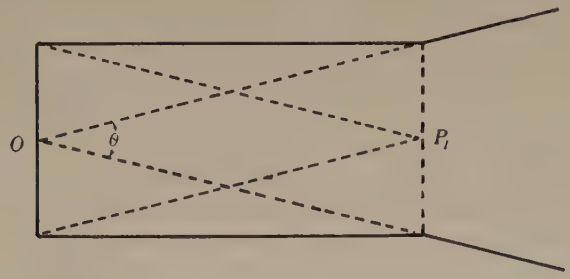

FIG. 10·6.

piston surface viewed edgewise. The radiation is "oscillatory" in nature out to P_1 where $OP_1 = m^2\lambda$. From that point on approximately there is conical divergence. We get a rough estimate of the extent of the divergence from the conical angle θ, which from the figure is seen to have the value

$$\theta = \frac{\pi a^2}{m^4\lambda^2} = \frac{\pi m^2 \lambda^2}{m^4 \lambda^2} = \frac{\pi}{m^2} \cdot \qquad (10{\cdot}51)$$

It thus appears, as we should expect from simple physical considerations, that for a given wave length the divergence varies inversely as the area of the piston surface.

As an illustration consider a plate for which $a = 12$ cm and vibrating with a frequency of 50,000 cycles. Then $\lambda = 2.90$ cm and $m = 12/2.9 = 4.14$, whence the radiation remains within a conical region of solid angle given roughly by $\pi/18$ steradians. The angle

[1] The above treatment is essentially the same as that of Crandall, loc. cit., p. 137 ff.

δ in the plane figure (Fig. 10·7) then comes out $\delta = 2$ arc tan $a/m^2\lambda$ $= 26°$ approximately. The experimental studies indicate that even smaller angles are found in practice than the above theory predicts.

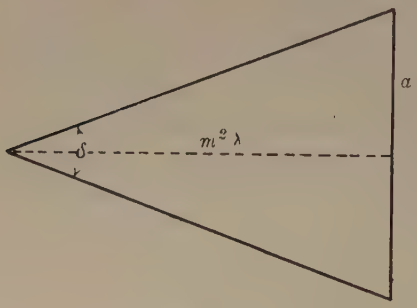

Fig. 10·7.

A more exact treatment of the problem of supersonic radiation involves the calculation of the intensity at distant points *off* the axis of the oscillator.[1] The problem is mathematically analogous to the Fraunhofer diffraction of light through a circular aperture. This leads to the result that if we consider the first sound diffraction ring on a plane at considerable distance from the source the lateral spreading of the radiation is confined to a plane angle of magnitude $\delta = 2$ arc tan $(.61\lambda/a)$. The analysis of the paper just mentioned shows clearly that contrary to what might be supposed from an uncritical interpretation of the approximate theory above there is really no parallel beam of sound of cross sectional area equal to the area of the oscillator at any distance from the source greater than its diameter. Nevertheless most of the sound is confined to a relatively narrow conical region extending outward with its apex at the oscillator. In fact at any distance from the source greater than, say, twice the diameter, the circle at the circumference of which the intensity falls to one-tenth of the intensity on the axis in the same plane will subtend at the center of the source a solid angle of magnitude $\pi(.45\lambda/a)^2$ steradians, corresponding to a plane angular spread of $\delta = 2$ arc tan $(.45\lambda/a)$. For practical purposes this result is probably as useful as that given by the Fraunhofer diffraction formula. Thus for an oscillator with $a = 10$ cm emitting radiation of frequency 50,000 cycles we get $\delta = 15°$ approximately.

10·9. Supersonics. The Piezo-electric Oscillator.—It will be of value to review briefly at this place the disadvantages in low frequency signalling which have stimulated research into the use of supersonic transmission. In the first place, the fact that the velocity of sound in water is nearly five times the velocity in air necessitates the use of large scale apparatus. For a wave of fre-

[1] See R. B. Lindsay, Phys. Rev., 32, 515, 1928.

quency 1000 cycles the wave length in air is 33 cm while in water it is 145 cm. But the best results (i.e., least scattering) are obtained with the use of apparatus of dimensions large compared with the wave length. The impracticability of the latter course is at once rendered evident. In the second place, there exists always in practice the difficulty of screening out the extraneous ship noises in receiving sound signals. These extraneous noises are usually of low frequency and according to report often interfere very materially with the clear reception of signals. In the third place, as we have had occasion to note above (Sec. 10·6), the radiation efficiency of any sound producing device at low frequencies is low. In each of these cases the increase in frequency helps to overcome the disadvantage and in addition serves, as we have noted in the preceding section, to concentrate the sound energy into a smaller region.

During the latter years of the great War (1917) the French physicist Langevin devised an acoustic oscillator with a frequency as high as 50,000 cycles, which since its frequency is outside the auditory range is generally referred to as a "supersonic" oscillator. In constructing this instrument use was made of the piezo-electric effect.

It had long been known that certain asymmetric crystals when subjected to stress become electrically polarized.[1] Conversely if in such a crystal polarization is produced by electrical means, it is accompanied by dilatation or contraction. Examples of crystals showing this effect are tourmaline, for which the discovery was first made, quartz and Rochelle salt or sodium-potassium tartrate. The latter shows the phenomenon in a most marked degree, but because of its more suitable mechanical qualities, quartz has been so far most extensively used in practical applications. Consider a section of a quartz crystal cut perpendicular to the optic axis (Fig. 10·8). The resulting cross section is roughly hexagonal. The optic axis may be thought of as extending out normal to the plane of the section at *O*. The lines *AB*, *CD*, and *EF* are the so-called electric axes, since along them the piezo-electric effect is most marked. For practical use it is customary to cut out of the crystal a slab of thickness *d* with sides parallel to the optic axis, as indicated by the dotted lines in the figure. If this slab is then inserted be-

[1] See, for example, Glazebrook, Dictionary of Applied Physics, Vol. II, p. 598.

tween two metal plates PP' (as in Fig. 10·9) (without necessarily having the quartz touch the plates), the plates being connected to an A.C. circuit, the oscillator thus formed will give rise to mechanical vibrations of the quartz plate with the frequency of the driving e.m.f. Conversely, if the quartz is made to vibrate through the

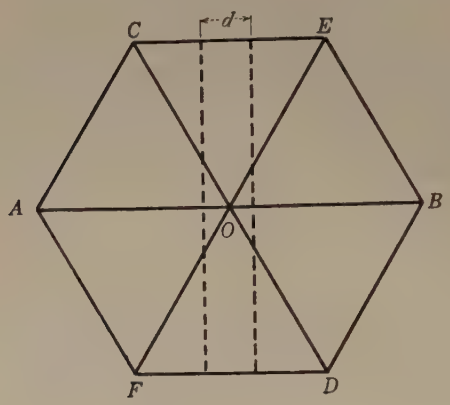

Fig. 10·8.

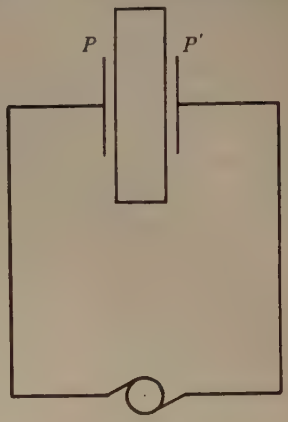

Fig. 10·9.

application of external mechanical pressure an alternating current will be set up in the circuit. There will be resonance when the frequency of the current is adjusted to equality with some one of the natural frequencies of the quartz crystal. This resonance is found to be extremely sharp. Experiments by Cady[1] with quartz oscillators of natural frequency about 90,000 cycles indicate a falling off of the current to one-half resonance value in an interval of 50 cycles either side of the resonance frequency, that is for a change of only .05%. It is this property which has made the use of the quartz oscillator so valuable in calibrating wave meters for radio work. The influence of the sharpness of resonance on the use of the quartz oscillator for signalling is noteworthy. For example, in the report of Langevin's[2] experiments, for which unfortunately not all the data are available in the literature, it is stated that using a piece of quartz *not* cut for resonance an applied potential of 50,000 volts was necessary to drive the oscillator so as to have it radiate 1 watt/cm^2 at a frequency of 40,000 cycles. For quartz cut so as to be

[1] Cady, Proc. Institute of Radio Eng., **10**, 83, 1922.

[2] See, for example, the article on Echo Sounding, Nature, **115**, 689, 1925.

in resonance at 40,000 cycles the applied potential for the above power output is only 1250 volts.

As has been intimated the natural frequencies of a quartz oscillator will depend on the way in which it is cut from the crystal. A section cut as in the above discussion with faces parallel to the optic axis will have three fundamental frequencies of which the highest is in the direction of its thickness. In general the highest fundamental frequency [1] is given by the empirical formula

$$\nu = \frac{287}{d} \times 10^3 \qquad (10\cdot52)$$

in cycles per sec where d is the thickness of the specimen in cm. For example, a frequency of 40,000 cycles will correspond to a thickness of 7.2 cm. As a matter of fact the oscillators in use in signalling are not single plates but mosaics of quartz imbedded in an insulating medium such as resin. Moreover the lower fundamental frequencies may be used so as to decrease the thickness of the pieces which are employed.

A typical arrangement as used by Langevin [2] is indicated in the accompanying figure (Fig. 10·10). In the sketch SS' represents an outline of a portion of the ship skin. The oscillator is Q with the

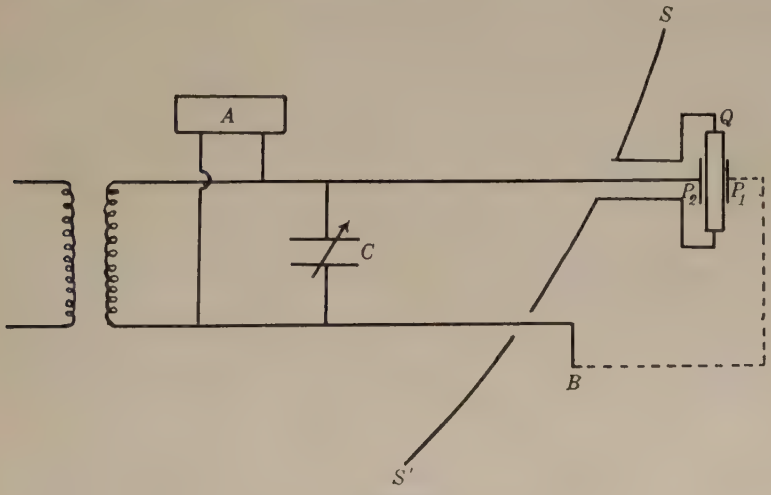

FIG. 10·10.

[1] See A. Hund, Proc. Inst. of Radio Eng., 14, 447, 1926.
[2] Taken from F. Collin, Le Génie Civil, 86, 38–41, 1925.

quartz mosaic in the center and the two metal plates P_1 and P_2 on each side. P_2 is connected to one side of an oscillating circuit wherein the frequency of the oscillations is controlled by the variable condenser C. The circuit of the oscillator is completed through the water from B to P_1. The source of high frequency oscillations for driving the oscillator is not shown; it is the usual arrangement employing the vacuum tube as generator. The part denoted by A represents a receiver-amplifier unit which is cut out while signals are being sent but which may be inserted for the reception of signals when the quartz oscillator is used as receiver. The frequency may be modulated by a beat method so as to render the incoming signals audible or an oscillograph of high sensitivity may be used to give a visible record which can be recorded on a moving photographic film. Details with regard to the practical application of these various schemes may be found in the article of Collin above referred to.[1] Much work on the piezo-electric oscillators is now being done in the United States but very little information concerning progress is available in the general literature.[2]

Brief attention must be paid at this place to the recent interesting experiments of Wood and Loomis [3] on high frequency sound waves of large intensity. Their work appears to open up a large field for future investigation of the physical, chemical and biological effects of this type of radiation. They have used acoustic waves generated in an oil bath by a piezo-electric quartz oscillator operating at 50,000 volts and at frequencies in the neighborhood of 300,000 cycles, the method of production being essentially the same as that of Langevin above discussed. The intensity is so great as to produce enormous radiation pressures, in one case amounting to about 3000 dynes/cm^2. They found that the waves can be transmitted along fine tubes and rods which when squeezed by the fingers will develop heat enough to produce painful effects. When the vibrations are communicated to liquids more or less stable emulsions have been formed, even of mercury in water. Numerous experiments were tried in the production of standing waves in plates and rods with such good success that wave length and

[1] See also, F. Collin, Le Génie Civil, **86**, 64, 1925.

[2] For recent theoretical investigations see A. Meissner, *Die Naturwissenschaften*, **17**, 25, 1929 (January, 1929). Also *Phys. Zeits.*, **28**, 621, 1927.

[3] R. W. Wood and A. L. Loomis, Phil. Mag., **4**, 417, 1927. See also F. L. Hopwood, Jour. Sci. Inst., VI, 34, 1929.

velocity measurements became practicable. For details as to the chemical and biological effects the original article should be consulted.

It may also be noted that the use of high frequency waves for the measurement of the velocity of sound in liquids has been successfully made by Loomis and Hubbard.[1] They have devised for this purpose a special type of sonic interferometer. A quartz oscillator operating in the neighborhood of 500,000 cycles produces standing waves in a column of liquid whose length can be very accurately controlled by a fine micrometer screw displacing a piston at the other end of the tube. As the length of the column is varied and the tuning passes through resonance points, characteristic reactions are produced in the oscillating circuit driving the quartz crystal, and these are used to measure the half wave length of the stationary waves in the liquid. This method has been developed to a very high degree of precision and the attempt has been made to apply it to the measurement of the velocity of sound in gases as well as liquids.[2]

In the same connection may be mentioned the result of experiments by R. W. Boyle [3] that there appears to be no change in the velocity of sound with the frequency in liquids over a range from 30,000 cycles to 600,000 cycles.

10·10. Acoustic Detectors. General Considerations.—Among low frequency detectors of sound signals in water we have to distinguish between the purely acoustic type into which no electrical connections enter and the acoustic-electric variety wherein sound vibrations are converted into electrical oscillations and then back into sound by means of the telephone receiver. In this section and the one following we are concerned with the former type only.

Acoustic detectors may themselves be divided into two main groups, viz., (1) *pressure* receivers and (2) *displacement* receivers. Receivers of the first class are operated primarily by the excess pressure produced by the sound in the medium in which the detector is placed. The *amplitude* of the vibration (of a membrane, say) is thus in this case a measure of the incident pressure. In the

[1] Jl. Opt. Soc. Amer., **17**, 295, 1928. Phil. Mag., **5**, 1177, 1928. The work of G. W. Pierce on the velocity of sound in gases at high frequencies (Am. Acad. Sci., **60**, 271, 1925) has already been referred to in Section 1·13.

[2] J. C. Hubbard, Phys. Rev., **35**, 1442, 1930.

[3] Nature, **120**, 476, 1927.

case of the second group the receiver measures directly the displacement of the medium.

Illustrations may serve to make the distinction clearer. Thus a diaphragm detector (e.g., a stethoscope) operates mainly on the pressure principle, while the "light body" discussed in Section 10·4 is a good example of the displacement detector, for it vibrates with a displacement amplitude of the same order of magnitude as that of the sound wave which strikes it. It must be emphasized, of course, that no real receiver functions exclusively according to either principle. For the exclusively pressure type instrument would necessarily be composed of a substance of infinite specific acoustic resistance, i.e., $\rho_0 c = \infty$, for only in this case would its displacement be zero, while the exclusively displacement type would conversely be made of a substance of zero specific acoustic resistance. Any practical receiver therefore uses elements of both types. Nevertheless it will, in general, operate chiefly according to one and the classification is therefore practical. It may be, moreover, of some importance in practice.[1] For let us suppose that we are to detect two sounds an octave apart in frequency with the same detector. Assume the latter is primarily a pressure detector and that its response is the same for both sounds. Since the dependence of intensity on pressure assuming plane or spherical waves as we do here is of the form

$$I = \overline{p^2}/\rho_0 c \tag{10·53}$$

we must conclude that the intensity of both signals is the same. Now the dependence of the intensity on the displacement amplitude is of the form (see eq. (1·47) Sec. 1·15)

$$I = \tfrac{1}{2}\rho_0 c\omega^2 A^2, \tag{10·54}$$

where A is the amplitude and $\omega = 2\pi$ times the frequency, as usual. Hence if the detector in question is primarily a displacement detector (i.e., measures A) the same response to the two sounds would *not* mean that they are of the same intensity, for the frequency of one is twice the other and the corresponding intensity is four times as great.

In general we may say that a receiver acts like a pressure detector when its specific acoustic resistance is very high compared with that of the medium in which it is placed, while it acts as a displace-

[1] See Aigner, loc. cit., p. 202.

ment receiver when its resistance is low compared with that of the medium in which it is placed. For in the former case its displacement due to the sound waves in the medium will be negligible and the pressure exerted on it is definitely measurable as in the stationary microphone, while in the latter case its displacement is greater than that of the medium and the displacement is the quantity measured. Since the specific acoustic resistance for spherical waves is a function of the frequency (see Sec. 3·2) the behavior of a given receiver will therefore in general depend on the frequency. But of course the waves here considered are practically plane, or in other words, the variation of specific acoustic resistance with frequency may be neglected for large values of r.

10·11. Acoustic Detectors. The Broca Tube.—The earliest type of acoustic receiver was the so-called Broca tube, consisting (see Fig. 10·11) of a sphere or nipple C of rubber or sheet metal attached to the end of a listening tube T. Such an instrument is primarily a pressure detector. It is still in use for submarine signalling. It embodies of course the stethoscope principle, but there is one important point of difference, namely that the rubber sphere or shell separating the water on the outside from the air of the chamber does not vibrate as a true medium but rather as a whole. We can not then apply at once our stethoscope formulae, but must discuss the vibrations of the shell. The theory of this type of receiver has been worked out in detail by H. A. Wilson.[1] We shall not consider this work in extenso, but give the principal results. It is evident that, irrespective of the shape of the receiver chamber, the incidence of a sound wave on the rubber nipple will cause variations in pressure and volume of the air in the chamber which will then be communicated to the air in the ear tube.

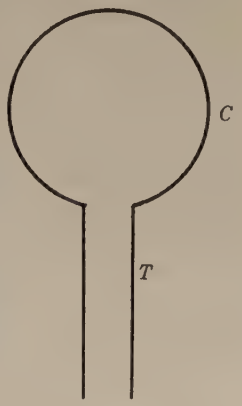

FIG. 10·11.

The influence of the magnitude of the volume of the nipple can be rather easily estimated. Thus let the equilibrium volume be V_0 and denote the change in volume corresponding to change in pressure δp by δV. Since the vibrations are adiabatic (see Sec. 1·13) we have

[1] H. A. Wilson, Phys. Rev., 15, 178, 1920.

$$\delta p = -\gamma p_0 \frac{\delta V}{V_0}, \tag{10·55}$$

where p_0 is the equilibrium pressure (atmospheric) in the chamber and γ is, as usual, the ratio of the specific heat at constant pressure to that at constant volume. The volume variation will be due to a combination of two effects, namely the vibration of the nipple and the flow of air out of the chamber into the tube. We write for the change due to the former $(\delta V)_1$ and have

$$(\delta V)_1 = Be^{i\omega t}, \tag{10·56}$$

where $B = (\delta V)_{1\,\max}$. The displacement in the tube will be given by

$$\xi = Ae^{i(\omega t - kx)}, \tag{10·57}$$

where $k = \omega/c$, as usual, and c is the velocity of sound in air. If the cross sectional area of the tube is S and if we take $x = 0$ at the junction of tube and chamber we have for the total volume variation of the latter

$$\delta V = (\delta V)_1 + S\xi_{x=0},$$

which becomes

$$\delta V = (B + SA)e^{i\omega t}. \tag{10·58}$$

Now the pressure variation at the junction must be the same for the air in the chamber as for the air in the tube (continuity of pressure). Hence we can write

$$\delta p = -\gamma p_0 \left(\frac{\partial \xi}{\partial x}\right)_{x=0} \tag{10·59}$$

or from (10·55)

$$\delta V = V_0 \left(\frac{\partial \xi}{\partial x}\right)_{x=0} = -iAV_0 k e^{i\omega t}.$$

Combining this with eq. (10·58) above, we have finally

$$A = -\frac{B(S - ikV_0)}{S^2 + k^2 V_0^2} = \frac{Be^{i\theta}}{\sqrt{S^2 + k^2 V_0^2}}, \tag{10·60}$$

where $\tan\theta = -kV_0/S$. Now the intensity of the sound in the tube is given by

$$I = \tfrac{1}{2}\rho_0 c\omega^2 |A|^2,$$

which in the present case thus becomes

$$I = \frac{1}{2}\rho_0 c\omega^2 \frac{B^2}{S^2 + k^2 V_0^2}. \tag{10·61}$$

This expression indicates clearly that to get the greatest intensity for a given signal and for given S, the volume of the chamber V_0 should be made as small as possible. It might be thought that the greatest effect would be obtained for $V_0 = 0$ exactly. But this would mean an infinite condensation at the end of the tube, an impossibility. Moreover, we are wholly neglecting viscosity which plays a greater rôle as V_0 gets smaller. For V_0 fixed, a smaller S contributes to larger intensity (stethoscope effect).

We need not discuss the problem in greater detail from a mathematical point of view. The interested reader will find the theory amply set forth in the article of Wilson just referred to. The general assumptions made are as follows: the nipple is in the form of a sphere and the forces that act on it are three in number, viz., (1) the force due to the incident sound wave from the water, (2) the reaction force due to the radiation of sound from the sphere back into the water, (3) the force due to the excess pressure in the air tube. The air in the nipple is assumed to act like an incompressible fluid. After writing down the equation of motion of the nipple, it is possible to deduce an expression for the velocity potential at any point in the air tube in terms of the incident sound pressure, and hence to get the power transmission up the tube to the ear. It is found that there is a resonance frequency for which the latter is a maximum, and moreover that there is an *optimum* cross sectional area of the air tube. As this area increases up to its optimum value the response becomes greater and the resonance sharper. For areas greater than the optimum the resonance becomes sharper but the response falls off greatly. Other conclusions of the theory follow in summarized form:

1. For a fixed value of the cross sectional area of the air tube the power transmission falls off as the frequency increases.

2. The intensity, other things being equal, varies inversely with the volume of the receiver chamber (see first part of this section).

3. In general, small surface area of the receiver chamber, other things remaining constant, leads to corresponding sharpness of resonance.

Extensive experiments on sound receivers of the Broca tube type have confirmed these theoretical conclusions qualitatively in every respect, though it would be too much to expect quantitative agreement in view of the necessary approximations involved.

10·12. Microphones for Subaqueous Reception.—Of acoustic-electric sound receiving devices there are four principal types, viz.: (1) microphone; (2) electromagnetic receiver (telephone type); (3) electrodynamic receiver (of which the Fessenden oscillator is an example); (4) condenser transmitter. All these may be used to transform sound vibrations into electrical oscillations.[1] Of these the microphone has so far proved most promising for subaqueous reception. It is this instrument which will be discussed in this section. It will be our endeavor to stress general principles in microphone construction rather than to present detailed descriptions of various types of microphones now available.

Microphones may be divided into two classes, corresponding more or less accurately to the general subdivision of sound receivers, into pressure and displacement detectors. The older type of microphone in which sound pressure produces variation in the resistance of an electrical circuit by varying the compression of an aggregate of carbon granules, belongs to the first class. It has not yet been designed for submarine signalling purposes. One of the most stringent requirements of a microphone for subaqueous use is that the sensitivity shall remain constant. This, however, is impossible with the pressure type microphone as ordinarily constructed since the slightest change in the static pressure (brought about, for example, by slight and unavoidable changes in the depth or by currents) will alter the sensitivity and throw the instrument out of adjustment. The newer type of microphone, the so-called "button" variety (*Shüttelmikrophon* of the Germans) is primarily a displacement detector. Its essential features are shown in the following diagram (Fig. 10·12). The whole instrument is contained in a sheathing or housing S, the front of which is in the form of diaphragm A. Securely fastened to the inside of the diaphragm by the coupling B is the microphone button. One electrode of the button is attached to B through a second diaphragm C while between the two electrodes are the carbon granules G. When a sound wave is incident on the diaphragm A, it shakes the coupling B causing a relative motion between D and E, for E and the housing remain

[1] One may be confused by the not altogether logical use of "receiver" and "transmitter." The terms have been used for many years in the telephone industry to refer to the *electrical* performance rather than the acoustic. A telephone receiver receives the current produced by the transmitter. The latter is, from the acoustical view, a receiver. In this chapter we will use the term "receiver" when acoustic reception occurs.

practically stationary because of inertia. The vibration disturbs the granules and so produces the desired alteration in resistance.

To understand the action of such a microphone it is very desirable to consider for a moment some ideas concerning the vibrations of *coupled systems*. In most problems in mechanical vibrations the mass factor and the elasticity or stiffness factor are never completely differentiated. For example, in the vibration of a diaphragm both the mass and the stiffness are distributed throughout the membrane. This mixture of elements is much more marked in the usual mechanical than in the electrical oscillations where capacity and inductance are fairly well separated in most oscillating circuits (neglecting distributed capacity which is generally small compared with the condenser capacity). So far in our study of vibrations in this book we have encountered only one illustration of a definite separation of the resonance elements of an acoustical vibrator. This is the Helmholtz resonator. It will be recalled (see Sec. 2·3) that the air in the opening moves as a whole providing practically all the inertia of the system while the stiffness resides almost entirely in the air of the chamber. But even here the separation is an assumption that is closely approximate, rather than completely real.

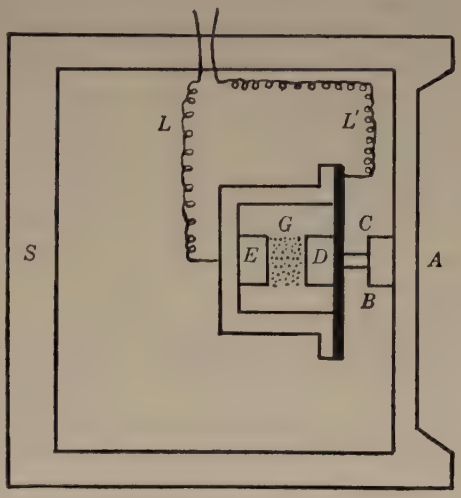

FIG. 10·12.

Nevertheless it is possible in an ideal way and probably will be of value to consider practical mechanical-acoustical vibrating systems as constructed from a certain simple system, namely two masses m_1 and m_2 joined by a massless elastic connection, as in the accompanying diagram (Fig. 10·13).[1] For low frequency oscillations, for example, this connector may be a very light spring, while for higher frequencies it must be thought of as a very light but rigid rod

[1] Hahnemann and Hecht, Phys. Zeits., 21, 187, 1920.

possessing a large elasticity coefficient compared with its mass. If the phase of the vibration is to be the same at all points of the rod, its length should, of course, be chosen much less than $\lambda/4$ where λ is the wave length in the rod of the longitudinal oscillations developed. Hahnemann and Hecht have constructed such vibrators and have

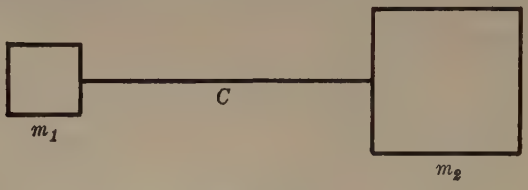

FIG. 10·13.

found that the resulting vibrations are as pure and undamped as those of a tuning fork. Such a system has been called in German a "Tonpilz." We shall call it here simply a standard coupled system. Often in practice one of the component masses is so large compared with the other that it may be safely neglected, as for example, when a spring vibrates with one end clamped in a vise.[1]

We shall get a clearer picture of the behavior of the coupled system considered if we solve its equations of motion, which fortunately are very simple if the mass of the connector is neglected.

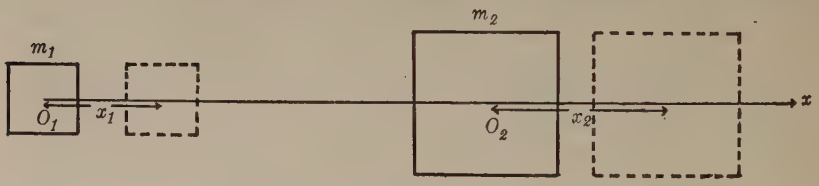

FIG. 10·14.

The displacement of m_1 from its equilibrium position O_1 will be called x_1. Similarly the displacement of m_2 will be measured from O_2 and designated by x_2. The positive direction is taken from left to right. The restoring force on m_1 may be written as $f(x_2 - x_1)$

[1] It may be objected that in no practical vibrating system is the differentiation of the elements sufficiently exact to allow the rigorous replacement by a standard coupled system as here defined. Moreover there exists also the difficulty that a coupled system practically equivalent for one range of frequencies may not be so for another. The authors recognize that the replacement is ideal but believe it will help in interpretation if used cautiously.

while that on m_2 is $f(x_1 - x_2)$, where f is the stiffness of the connector. The equations of motion then are

$$\begin{aligned} m_1\ddot{x}_1 + f(x_1 - x_2) &= 0, \\ m_2\ddot{x}_2 + f(x_2 - x_1) &= 0. \end{aligned} \tag{10·62}$$

Adding the two we get

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0. \tag{10·63}$$

If we denote the displacement-amplitudes of the two masses respectively by a_1 and a_2 and assume that they perform simple harmonic motions in the same phase we have then

$$m_1a_1 + m_2a_2 = 0, \tag{10·64}$$

whence the amplitudes are inversely proportional to the masses and oppositely directed. That the same is true of the individual kinetic energies is seen from the relations

$$\frac{E_1}{E_2} = \frac{m_1a_1^2}{m_2a_2^2} = \frac{m_2}{m_1}. \tag{10·65}$$

Thus if one of the masses is much smaller than the other practically all the kinetic energy of the system is concentrated in it. To damp the motion, then, the damping force should be applied to the smaller mass. For the same reason to avoid damping one of the masses should be made as large as possible so as to give rigidity to the system, for the heavier it is the less energy it will absorb from the motion of the smaller mass. It should be emphasized once more that we are here assuming that all parts of the connecting rod vibrate in the same phase, i.e., there is *no* wave motion or phase difference along the rod. If eq. (10·63) is integrated, assuming simple harmonic motion and equality of phase (i.e., each displacement varying as $e^{i\omega t}$), we have

$$x_2 = -\frac{m_1x_1}{m_2}.$$

If this is substituted in (10·62), we obtain

$$m_1\ddot{x}_1 = -f\left(1 + \frac{m_1}{m_2}\right)x_1 = -m_1\omega^2x_1, \tag{10·66}$$

whence the frequency comes out to be

$$\frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{f(m_1 + m_2)}{m_1m_2}}. \tag{10·67}$$

To get high frequencies (i.e., in the acoustic range) we must either make f very large or m_1 and m_2 very small. The former is the more desirable as one often wishes to use heavy masses. Introducing Young's modulus Y we have $f = YS/l$, where S is the area of cross section of the rod and l is its length. The rod must then be as short as possible with a high modulus. At the same time its mass should be small. If we denote the latter by m_s and suppose it small compared with m_1 and m_2, it may be shown that the correct value of the frequency is obtained by substituting into eq. (10·67)

$$\begin{aligned} m_1\left[1 + \frac{m_s}{3m_1}\cdot\frac{m_2}{m_1+m_2}\right], \\ m_2\left[1 + \frac{m_s}{3m_2}\cdot\frac{m_1}{m_1+m_2}\right], \end{aligned} \qquad (10\cdot68)$$

for m_1 and m_2 respectively. It is to be noted, of course, that even though the mass of the connecting rod makes little change in the frequency of the system, it does introduce a new natural mode of vibration, viz., that of the rod itself. In general, however, it can be arranged to have the fundamental frequency of this vibration very high compared with that of the coupled system. Hence it will not trouble the latter materially.

The inverse relationship of amplitude and mass should make the standard coupled system of great value when a transfer of amplitudes is desired, that is when one wishes to produce a large vibration amplitude from a small one or conversely. In this case the system may be said to act like a lever. As an example we may consider the electromagnetically operated diaphragm as a source of sound waves in water. This is a rather inefficient source as will be seen from the following line of reasoning.[1] A sender placed about one meter under the water surface with a radial intensity of more than $\frac{1}{3}$ watt per cm^2 at the source produces bubble disturbances which break up the sound field. From the expression for the intensity $I = \frac{1}{2}\rho_0 c\omega^2 A^2$, it is seen that this at once puts a limit to the allowed amplitude for a given frequency. Thus for 1000 cycles we find that the limiting value of A is 10^{-3} cm, approximately. Now suppose the diaphragm is vibrating in resonance with the applied force. It then radiates maximum energy and we have simply

$$\text{Force} = R\dot{\xi},$$

[1] See F. Aigner, Unterwasserschalltechnik, pp. 129, 135, Berlin, 1922.

where R is the damping factor and ξ the displacement. If the mechanical damping is small enough to be neglected in comparison with the radiation damping, we then have (Sec. 10·6) for high frequencies, $R = \rho_0 c S$, where $S = \pi a^2$, the effective area of the diaphragm; while for low frequencies the corresponding quantity (eq. 3·27) is $R = \rho_0 \omega k S^2/2\pi$. Now the average rate of energy radiation is

$$\dot{W} = R\dot{\xi}^2_{\max}/2.$$

Under the conditions specified above it is found that $\dot{\xi}_{\max}$ should not exceed 2π cm/sec. Let us suppose we wish the sender to radiate 1 K.W. of power. Putting $\dot{W}$ equal to 10^{10} ergs/sec we find that $R = 5 \times 10^8$ c.g.s. units. Hence by the force equation, the maximum force must be $10^6\pi$ gm wt or of the order of *three tons weight.* With an electromagnetically operated diaphragm it is hopeless to attain such a maximum force at the frequency indicated except at a considerable loss in efficiency. It is here that an acoustic lever should be of great value. For if the vibrating diaphragm is connected to the smaller mass of a standard coupled system instead of radiating directly into the water, it will cause this mass to vibrate with a large amplitude. At the same time the vibrations are communicated to the water through the larger mass with larger radiating surface as waves of smaller amplitude corresponding to the same average energy dissipation but smaller intensity. The radiation is therefore far more efficient than if it took place directly from the diaphragm to the water. The same instrument could be used for reception, transforming the incident sound waves with their small amplitude into large amplitude vibrations of the diaphragm.

Possible difficulties in the way of the technical application of these theoretical considerations may be: (1) the effect of the mass of the vibrating diaphragm which will itself form a coupled system with the smaller mass mentioned, and (2) the difficulty of rendering the connection efficient.

Let us now discuss the carbon granule microphone from the standpoint of the fundamental coupled system. Diagrammatically we can represent the microphone as in the following sketch, Fig. 10·15. Thus the housing will represent the mass m_1 while m_2 is the effective mass of the outer membrane, the stiffness connecting the two being provided by the membrane itself. This constitutes one coupling. Coupled to it is another system of which the one mass is

made up of the mass m_2 plus the mass of the button diaphragm, while the other mass is m_3, that of the button housing. In this second system the elasticity is divided, part of it residing in the button diaphragm and the rest in the carbon granule filling. The latter, of course, also provides damping for the system.

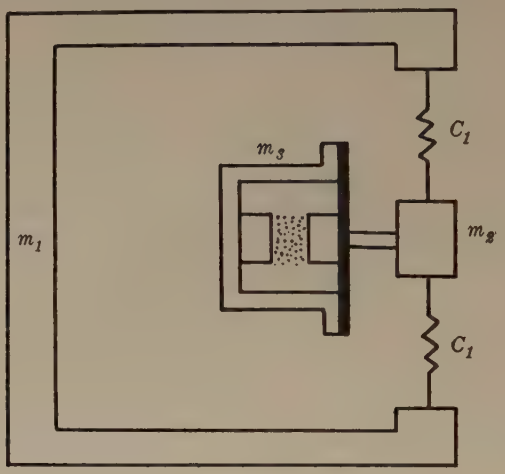

FIG. 10·15.

It should be noted that in this case the coupling between m_1 and m_2 is through a transverse vibration rather than the longitudinal vibration of the standard coupling previously described. Nevertheless in its general lines the main argument is still applicable.

Now in practice it has been found [1] that the tuning of a microphone for subaqueous work is very variable. For some time the reason for this was not clear. But an analysis of the instrument into its component parts as immediately above clarifies this difficulty. In most of the microphones previously used the larger part of the elasticity of the button resided in the granules. But this varies considerably with variations in the pressure on the granules thus accounting for the uncertain tuning. The solution obviously is to insert most of the elasticity into the button diaphragm. This might be accomplished by making the packing looser, though unfortunately this procedure is often accompanied by an increase in

[1] See Aigner, loc. cit., p. 208. It may be mentioned that most of the material in this section is taken from Aigner's book or from his original source, namely Hahnemann and Hecht.

the noisiness of the microphone when in use. Compromise must be reached through actual experiment to attain satisfactory adjustment.

Contrary to what might be supposed from their essential simplicity of design, microphones have given much trouble in practice. One difficulty has been that they do not attain their maximum sensitivity at once after the closing of the circuit. The period of growth may be as much as 30 seconds. The reason for this is doubtless to be found in the fact that the air in the microphone has to come to some kind of equilibrium during the process of expansion of the air and the various component parts of the instrument. This difficulty can be overcome only by the very careful choice of material for the construction of the microphone, for example, metals of small thermal expansion coefficient. Equalization of pressure by communication of the microphone chamber with the outside air also helps.

Other requisites for successful operation of microphones are: water-tight housing, proper size and shape of carbon granules, adjustment of the button support in order that it may vibrate with but one degree of freedom, appropriate resistance of the microphone circuit, which should be at least equal in magnitude to that of the microphone itself.

One other important thing may be taken up at this place because it enters so largely into the use of microphones for direction finding based on the binaural effect (see Sec. 10·15). This is the necessity of "matching" microphones for phase when several are to be used together. It will be recalled that when an external periodic force acts on a vibrating system, in order that the force shall contribute energy to the system it is essential that there be a difference in phase between the force and the resultant displacement (see Sec. 2·2). For frequency of the force $= \omega/2\pi$, this phase angle α is given by

$$\cot \alpha = \frac{m\omega^2 - f}{R\omega}, \qquad (10\cdot69)$$

where m, R and f are, as usual, the mass, resistance and stiffness factors of the system. If two or more microphones are matched for phase it means that the angle α for all of them for a given frequency is the same. The matching is usually done experimentally by actuating mechanically each pair of microphones and allowing the currents to oppose each other. If the resultant current is zero the microphones are assumed to be matched. There is a difficulty that

must be watched for in such a procedure. Examination of the formula for cot α shows that for ω far away from the resonance value two microphones may be fairly well matched even though there are slight differences in m, R and f for the two. These differences, however, will have a very pronounced effect in the immediate neighborhood of the resonance frequency for which $\alpha \doteq \pi/2$, resulting in lack of phase agreement between the two instruments. This may lead to serious error in the use of such instruments in lines of receivers for direction finding, to be discussed in a later section, and renders it essential that the matching be performed with very great care throughout the resonance range.

10·13. The Tunable Diaphragm.—During the late war, L. V. King[1] succeeded in developing a microphone receiver with a resonance frequency alterable over a considerable range. It is clear that this might possess considerable value in detecting from the frequency of the sound picked up the nature of the source. For this purpose a receiver with a fixed resonance frequency is practically useless. Briefly put, King's method consists in altering the air pressure in the microphone chamber, thus altering the tension in the diaphragm. The accompanying sketch gives the principle in diagrammatic form (Fig. 10·16). The apparatus differs little from

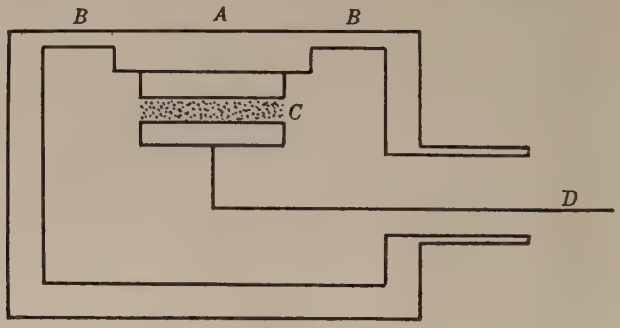

Fig. 10·16.

the conventional microphone save that the outer housing diaphragm is made with most of the mass concentrated at the center (A) while the annulus BB is rather thin. For example in a type case cited in King's paper, the thickness at A is .07 cm while that for the annulus is .03 cm. This is for an annulus of inner radius 1·75 cm and outer

[1] L. V. King, Proc. Roy. Soc., 99, 163, 1921.

radius 3.50 cm. *D* in the diagram denotes an air tube (through which the electrical connections may also go) by means of which the pressure in the chamber may be controlled. Increase of pressure above the atmospheric will increase the tension in the annulus and thus raise the frequency. Of course, a limit is provided by the elastic limit of the diaphragm material (nickel-chrome steel in King's experiments). The following graph (Fig. 10·17) indicates the order

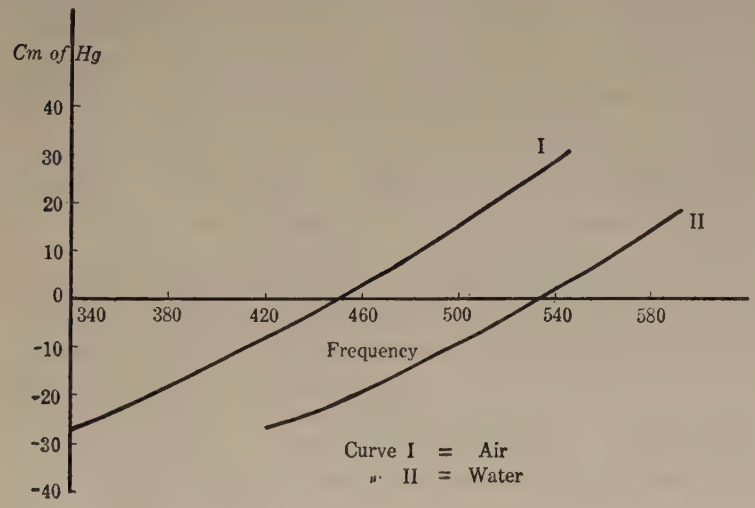

FIG. 10·17.

of magnitude of the results obtained. The ordinates represent excess or gauge pressure in cm of Hg while the abscissa measures frequency. This shows the effect over a rather small range. King says that a range from 450–1980 cycles is possible but gives no definite data.

10·14. Principles Used in Localization. The Binaural Phase Effect.—It is now necessary to indicate briefly the principles underlying the methods by which sound receivers may be used to determine the *direction* from which sound approaches the listener. The acoustic receiver (the Broca tube, as above discussed) has no directional property whatever. This is not true of the microphone. We have already noticed (Sec. 10·4) that a single microphone used in water is many times more sensitive to a sound approaching from directly ahead of the diaphragm than to the same sound approaching at right angles. The single microphone can then serve as a direction

finder but it is not a particularly accurate one for practical purposes without the use of a baffle plate the effect of which has been noted in Section 1·4.[1] The two most important principles now in use for direction finding are the so-called binaural phase effect and the maximum intensity effect. The first of these is the more important for signalling purposes and its application will be discussed in this section. The ability of an individual with two normally functioning ears to center a sound (in the absence of too many reflecting and scattering surfaces) by turning his head until the source of sound seems to lie in the median plane of the ears is undoubtedly one of the most remarkable of the many physiological capacities of man. Realization of this fundamental property of the auditory system doubtless came early but it was not until the middle of the last century that a scientific study of the matter was begun. The work has been somewhat complicated by the fact that there are really two possible influences at work. In the first place difference in *intensity* at the two ears undoubtedly has something to do with the ability to detect sound direction. On the other hand the difference in *phase* at the two ears of a sound from a given source also leads to an apparent displacement of the source from the median plane. One of the authors has made a careful study of both effects [2] with the definite conclusion, based upon quantitative measurements, that the binaural intensity effect is not competent to explain the ability to locate sound under all conditions, while the phase effect is, at any rate, up to frequencies of the order of 1500 cycles. Hence the latter effect is by far the more significant factor in direction finding. Stewart gives as his most convincing reason for this conclusion the fact that for most observers there are certain frequency ranges in which wide variations of intensity at the two ears seem to produce *no* displacement of the apparent source of sound (i.e., it stays fixed in the median plane). In these same "lapse" regions the phase effect, however, continues to be operative. The physiological and psychological aspects of the binaural effects are dealt with in Chapter Nine.

As far as the application of the binaural phase effect to the detection of the direction of a sound signal is concerned, consider the

[1] See also the experiments of Wood and Young, Proc. Roy. Soc., **100**, 261, 1921–22.

[2] G. W. Stewart, Physical Review, **9**, 502, 1917. In the three successive articles there presented there will also be found very complete references to the literature on the binaural effects. In the present volume there is a careful survey of the subject in Sec. 9·12.

accompanying figure (Fig. 10·18). Two acoustic receivers O and O' are placed at the ends of a rod of length l. The source of sound is, of course, at a distance very large compared with the length l. OE and $O'E'$ are the ear tubes. The path difference between the sound reaching O and that reaching O' is approximately

$$l \cos \varphi = l \sin \theta. \quad (10\cdot70)$$

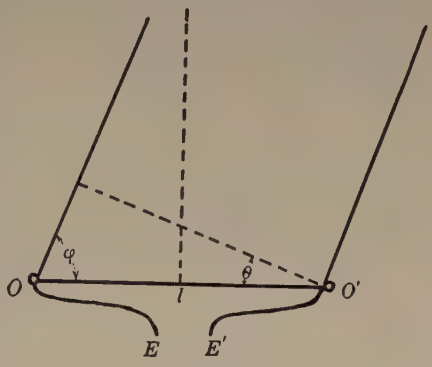

Fig. 10·18.

The corresponding *phase* difference is $k_1 l \sin \theta$ where $k_1 = 2\pi/\lambda_1$ and λ_1 is the wave length of the sound in the water. Rotation of the rod through the angle θ in the appropriate direction will reduce the phase difference at the ears to zero and the sound source will then appear to lie in the median plane represented in the figure by the dotted line perpendicular to OO'. The angle θ then measures the direction of the source from the observer.

In this connection it should be pointed out that if l is rather long it may be that there will be some angle or angles θ such that

$$l \sin \theta = n\lambda_1, \quad (10\cdot71)$$

where n is an integer. In this case there will be equality of phase at the two receivers and the sound will appear to come from straight ahead. To avoid this error it is well to make l rather short (and indeed this course is dictated by practical convenience as well) and of the order of the shortest wave length which the receiver is expected to pick up.

From the theory of H. A. Wilson [1] it would appear that the best value of l is equal to one-half the wave length in water for the resonance frequency of the receivers. The practical importance of this may not be very great owing to the fact that the resonance frequency is not particularly important.

10·15. Use of Compensation in Localization.—In the modern use of the binaural phase effect for submarine direction finding all

[1] H. A. Wilson, loc. cit., p. 186.

actual turning of the receiver system (e.g., the rod joining the two receivers in Sec. 10·14) is given up and replaced by the more convenient equivalent plan of introducing compensation in the acoustic line from the receivers to the ears. Thus to refer back to Fig. 10·18 of the preceding section, instead of rotating the rod, one can shorten the ear tube OE or lengthen the ear tube $O'E'$ by an amount such that the difference in length $l_2 - l_1$ is equal to

$$l_2 - l_1 = \frac{k_1 l \sin \theta}{k},$$

where $k = 2\pi/\lambda$ and λ is the wave length of the incident sound taken in air instead of water. Thus the direction is immediately given by

$$\sin \theta = \frac{c_1(l_2 - l_1)}{cl}, \qquad (10\cdot72)$$

where c and c_1 denote the velocity of sound in air and water respectively. In the so-called rectilinear oscillator the shortening or lengthening is performed by the use of a trombone adjustment. The instrument may be made to read θ or φ directly.

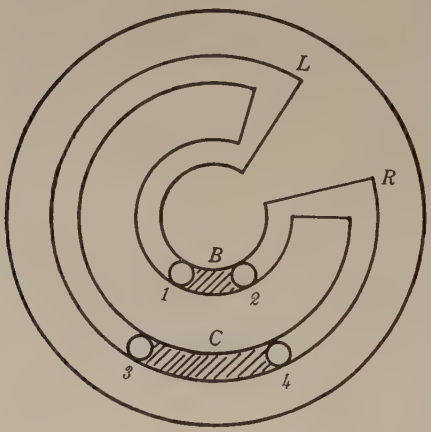

Fig. 10·19.

A more compact form of compensator than the simple rectilinear type has been developed in this country. It is represented graphically in the following diagram and consists of two grooves in a revolvable circular plate fitted to a stationary plate having in the grooves two snugly fitting blocks. These are represented in the figure by B and C. Tubes are inserted at 1, 2, 3, and 4. A tube from one of the receivers goes to 2; the sound then passes through the curved part R and out to the ear at 4. The other sound receiver is connected with the ear via 1 and 3 and the side L of the compensator. The rotating top can be calibrated to read directly the angle θ.

We must now note certain factors of error in the use of the

compensator. In the first place when the source of sound is nearly dead ahead only a slight change in the air path is necessary to bring the sound into the median plane and a small error in the compensation makes a large error in the resulting angle. In the second place there is generally present an ambiguity as to the actual direction. For to consult the figure (Fig. 10·20) it is seen that

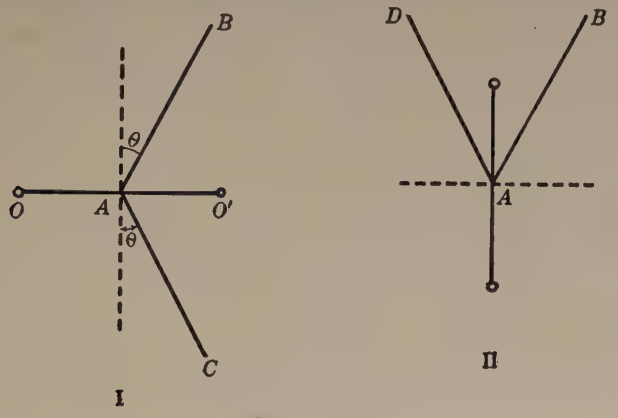

Fig. 10·20.

(from I) for a given setting θ the sound may be coming either in the direction BA or CA. This ambiguity may be removed by noting which way the apparent source moves with the alteration of the compensator. When it is displaced from its position of equality of phase the apparent source moves opposite to the rotation of the compensator if the actual source is in the direction AB and in the direction of the rotation of the compensator if in the direction AC. The ambiguity may also be overcome by rotating the receiver bar through 90° as in II. The two directions corresponding to the same setting here are DA and BA. The direction common to the two observations, namely, BA, is then the correct one. Obviously this the procedure may be carried out without the necessity of rotating bar by having three fixed receivers at the vertices of a triangle as in the accompanying figure (Fig. 10·21). Two observations taken in

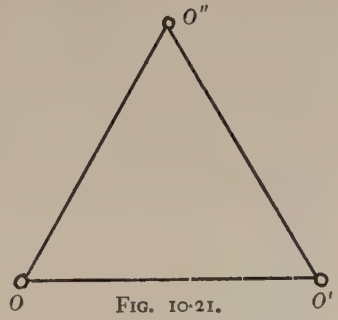

Fig. 10·21.

succession with two different pairs of receivers, as OO', $O'O''$ or $O''O$ will then suffice to fix the sound direction unambiguously. Four receivers placed at the corners of a square are also often thus employed, introducing the factor of safety that the failure of one of the receivers will not prevent the use of the instrument.

10·16. Multiple Receiver Systems.—Early in the study of acoustic receivers it was recognized that several receivers used together with the individual tubes leading into a large common tube might gather more sound than one alone. The theoretical treatment of such collections of receivers is extremely complicated. It has been worked out in detail by H. A. Wilson.[1] We shall not discuss the theory here.

In most practical multiple receiver systems the units are arranged in a line as in the figure (Fig. 10·22). If the tubes from each

Fig. 10·22.

receiver go to a common tube leading to the ear, then it is evident that, for sound coming from directly ahead or directly behind, the intensity at the ear will be a maximum. For sound coming from any other direction the sound reaching each receiver will be out of phase with that incident on any other and a certain amount of interference will ensue, diminishing the intensity at the ear. Thus such a line of receivers can be used to determine the direction of sound by what may be called the maximum intensity method. That is, the line of receivers may be rotated until the sound in the ear (the sound from all the receivers will in this case go to *both* ears) is a maximum. It is more common to dispense with the rotation and introduce variable compensation into the tubes leading from the receivers. Wilson shows that there is for the resonance frequency an optimum length l for the distance between adjacent receivers. This is given by

$$l = \frac{\rho_1 \omega_1 S}{4\rho_0 c}, \qquad (10\cdot73)$$

where S is the area of cross section of the individual tubes leading from the receiver units to the main ear tube. The resonance frequency is $\omega_1/2\pi$, while ρ_1 and ρ_0 are the densities of water and air

[1] H. A. Wilson, loc. cit.

respectively and c the velocity of sound in air. It is then shown that the area in the water across which there passes per second an amount of sound energy equal to that which crosses the main tube (assumed to be of cross section equal to the sum of the cross sectional areas of the individual tubes) is given by

$$A = \frac{\lambda_1(n-1)l}{2\pi}, \qquad (10{\cdot}74)$$

where n is the number of units, i.e., $(n-1)l$ equals the length of the line, and λ_1 is the wave length of the sound in water. Of course, we must be careful not to put too much emphasis on resonance conditions, as these are not the important ones in practice.

A great variety of multiple acoustic receivers have been constructed. For a description of the characteristics and uses of these reference should be made to articles by Harvey C. Hayes.[1] Most of these have been superseded by lines of microphone receivers to be discussed further on in this section. But one type, the so-called multiple forward (*MF*) tube, is of sufficient acoustic interest to be worth mentioning here. Two receivers (acoustic) are joined by a tube (see the accompanying figure) which is tapped at a point C

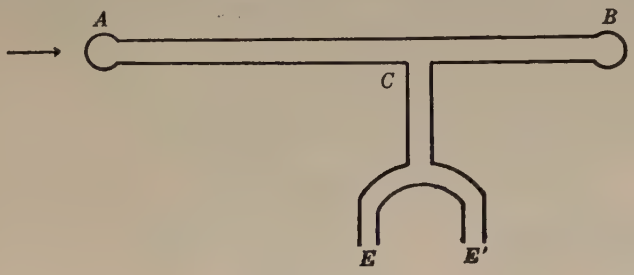

FIG. 10·23.

between them, and the sound therefrom led to the ears. Let us suppose that sound is approaching this pair of receivers from the direction of the arrow. The receiver A will then be first excited and the resultant disturbance will travel down the tube to the outlet C. At the same time the sound will travel through the water from A to B, excite B, and the resulting disturbance will travel through the tube from B to C. Now if the two disturbances reaching C from either side are in the *same* phase, reinforcement will ensue and the

[1] See in particular H. C. Hayes, Proc. Amer. Phil. Soc., **59**, 1, 1920.

sound emerging from C will be of maximum intensity. Any difference in phase will materially decrease the intensity. If we wish maximum intensity to result for sound approaching in the direction of the arrow we must insert the outlet C in such a way as to provide for phase agreement. Let $AC = x$ and $AB = l$. The condition for phase agreement then is

$$\frac{x}{\lambda_a} = \frac{l}{\lambda_w} + \frac{l - x}{\lambda_a},$$

where λ_a and λ_w refer to the wave lengths in air and water respectively. From this we get

$$x = \frac{cl}{2}\left(\frac{1}{c} + \frac{1}{c_1}\right), \qquad (10{\cdot}75)$$

where c = velocity of sound in air and c_1 = velocity of sound in water. Introducing the numerical values we have $x = .615l$, which fixes the position of the outlet. This multiple receiver is thus a maximum intensity direction finder, for when the intensity of the sound at the ears is a maximum the sound source must lie in the direction of AB and nearer A. As a matter of fact these receiver pairs have been used in groups of four (eight receivers in all) as indicated in the following diagram (Fig. 10·24).

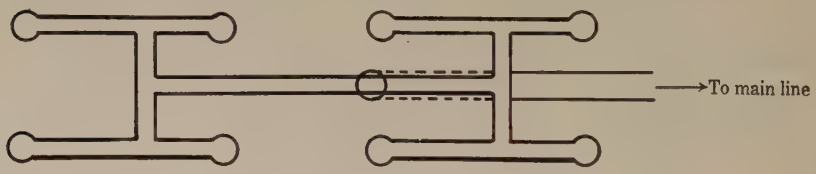

Fig. 10·24.

To illustrate how such receiver units may be combined into a line utilizing the binaural phase effect for direction finding, we may consider for a moment the so-called MV tube (multiple variable) (for a detailed account see Hayes, loc. cit.). This is now more or less superseded in this country and has been replaced by a similar instrument employing microphones instead of acoustic receivers. It is of interest, however, as representing the most elaborate use of the purely acoustic receiver. This instrument is made in two parts; each consists of a line of twelve receiver units, and each unit in

turn is of the type shown in Fig. 10·24. One line of twelve is usually placed on the starboard side and another on the port side of a vessel more or less parallel to the keel. As indicated in the figure, each line of twelve is subdivided into two groups of six each which connect to each ear separately, thus providing for the employment of the binaural effect. The compensation device is indicated schematically by the trombone-like attachments *a*, *c*, *d*, *f*, *g*, *i*, *j* and *l* (Fig. 10·25). The receivers 2, 5, 8, 11 are joined

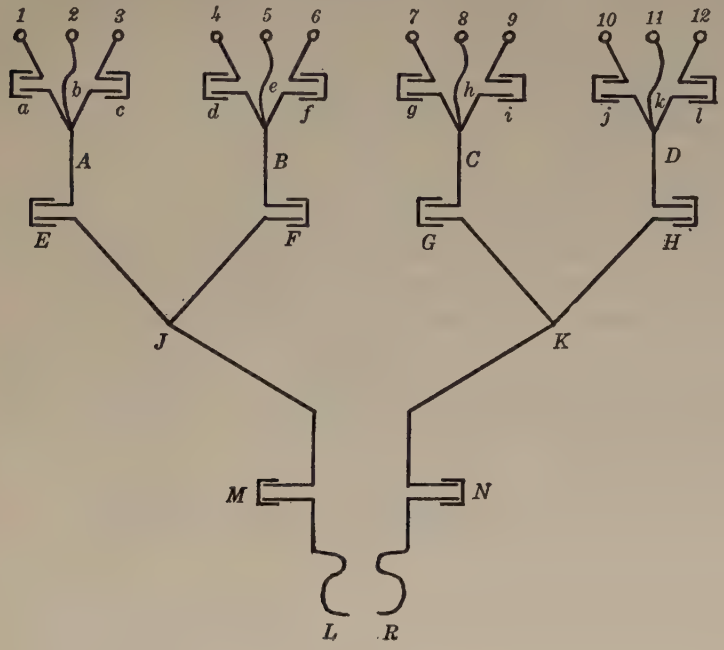

Fig. 10·25.

to *A*, *B*, *C*, *D* by tubes of invariable length. It has been possible to combine the whole compensation system into one circular compensator with appropriate grooves. Naturally the compensation introduced by each element must be made dependent on that introduced by the others and is usually of a magnitude equivalent to the phase difference corresponding to a certain small distance in water, say from 5 to 10 cm. The binaural base-line, so to speak, is provided by *JK*, the sound from *J* going to the left ear, and that from *K* to the right. In use it has been customary to get the

bearing on the sound by employing first the starboard line alone and then the port line alone. Finally the forward six receivers of each line are used binaurally in what is called "cross" compensation to check the previous determinations. This, of course, introduces the need for a special type of compensator.

The electrical *MV* tube is similar to the acoustical, save in that it uses microphones and necessarily employs electrical compensation. It is of interest to note briefly the essential features of this instrument, which is now in rather wide use. In place of the 12 acoustic receiver units in the type above discussed there are 12 button microphones constructed more or less on the lines laid down in Section 10·12. In some types there are eighteen microphones in a line instead of twelve. In either case half the microphones are connected to one ear phone and the rest to the other. The connection is made through transformers in the secondaries of which are placed retardation units for the compensation. The retardation lines are of the type known as capacity coupling [1] and are indicated in the following diagram (Fig. 10.26). This is an electrical

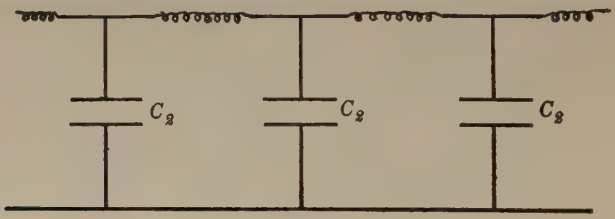

Fig. 10·26.

filter with a change of phase (retardation) from section to section, but with practically no attenuation. For a line of this kind Pierce shows that the time lag introduced per section is $\sqrt{L_1 C_2}$. This time lag per section is usually arranged to be equal to the time required for the sound to travel a certain small distance in water, say from 5 to 10 cm. For example, if it were 10 cm, the time $T = 10/14.5 \times 10^4$ sec $= 6.9 \times 10^{-5}$ sec. The compensation is accomplished by the insertion or removal of individual sections from the ear phone circuit. The above type of line shows only slight attenuation at reasonably low frequencies (1000 cycles or less).

The advantages of the electrical *MV* type of installation over the acoustical may be summarized as follows. First, the micro-

[1] See G. W. Pierce, Electric Oscillations, p. 320.

phone line may be strung along a cable which may be pulled in with comparative ease, rendering it possible to repair defective units without docking the vessel, which is necessary for the repair of the more rigidly installed acoustic units. Second, the compensator may be placed where it is most convenient, whereas the acoustic compensator must be placed as near the receivers as possible. This latter course is usually disadvantageous.

10·17. The Echo Method of Acoustic Depth Finding.—All modern acoustic methods for depth finding are based on the fact that whenever a sound signal is emitted by a subaqueous oscillator there is a reflected signal from the sea bottom. Use of this may be made in two ways. In the first place for depths up to about 100 fathoms it is customary to use the regular sound receiving line to get a bearing on the reflected sound (that originates, say, from the propellers), and the determination of the depth is then a problem in trigonometry. For depths over 100 fathoms this method is not accurate and in such cases the time for definite impulse signals (such as the intermittent sounds from a Fessenden oscillator) to travel to the bottom and return is measured. From this and a knowledge of the velocity of sound in sea water the depth may be ascertained.

The details of the measurement by the first method are indicated [1] in the accompanying diagram (Fig. 10·27). The water line

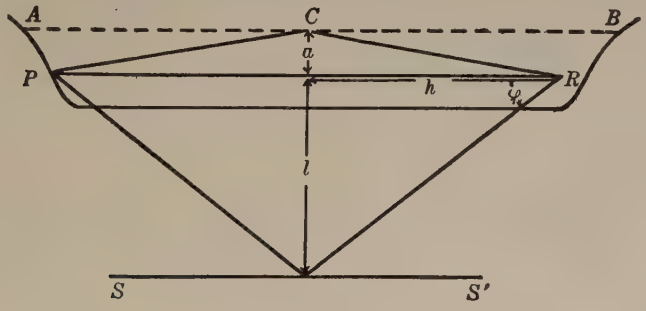

FIG. 10·27.

is denoted by AB. The line from the propeller to the receiver system is PR and is of length $2h$. If the bearing of the reflected sound indicated by the receiver system is φ, we then have

$$l = h \tan \varphi$$

[1] See H. C. Hayes, Proc. Amer. Phil. Soc., 59, 317, 1920; 63, 134, 1924.

and the depth is then given by

$$a + l = a + h \tan \varphi. \tag{10·76}$$

The receiver compensator can be calibrated to read this directly. When the bottom is sloping the determination is more complicated. Two separate receiver systems and two sound sources are necessary. Thus in the diagram (Fig. 10·28) a sound source and receiver are

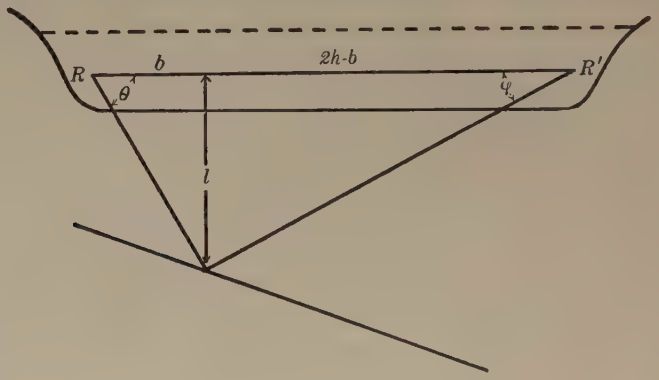

Fig. 10·28.

placed at both R and R'. A bearing is taken at R on the sound from R'. Let the corresponding angle be θ. Similarly, a bearing is taken at R' on the sound from R. Let the angle be φ. Then from simple trigonometry we have for the depth

$$l + a = a + \frac{2h \tan \theta \tan \varphi}{\tan \theta + \tan \varphi}. \tag{10·77}$$

In both cases the question may be raised concerning the wave reflected at the water surface. But we have already noted that for sound reflected in the act of passing from water to air there is a change of phase. Hence (as in Fig. 10·27) the reflected sound traveling by the path PCR will interfere destructively with the sound traveling by the direct path PR. Thus the bearing at R always is on the sound reflected from the bottom and not the surface.

In order to use the second method indicated in the first paragraph of this section it is necessary to develop a very accurate and at the same time rigidly constructed time measuring instrument. This has been done by the U. S. Navy in the so-called sonic depth finder.

The principle on which it operates is indicated in the following figure (Fig. 10·29). On a uniformly rotating horizontal disc AB covered with a good friction surface there bears a vertical wheel CD of smaller dimensions. This rotates by the friction between its edge and the surface of the horizontal disc. Its rotational velocity depends, of course, directly on its distance from the center of AB, and this distance can be varied by advancing or retarding the shaft on which it rotates by means of a screw. At each revolution of this wheel a sharp sound signal is sent out by a Fessenden oscillator.

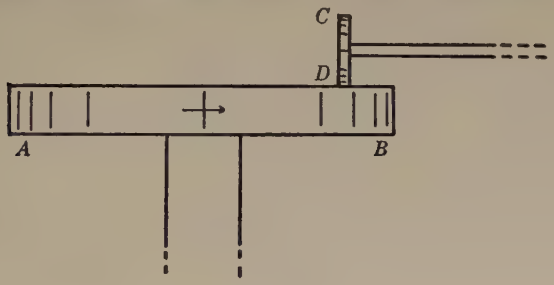

FIG. 10·29.

One ear phone of a head set is connected to this intermittent source of sound while the other is connected to the sound receiver which picks up the reflected signal from the bottom. It is evident that if and only if the wheel CD revolves an integral number of times while the sound from a given signal is traveling to the bottom and back, the responses in both ears will be simultaneous. Suppose coincidence has been brought about when the time between outgoing signals is $t_1 = RT/g$, where R = radius of the small wheel, g = the distance of the small wheel from the center of the horizontal wheel, and T = period of revolution of the latter. If we represent the depth by l we then have

$$2l = \frac{pc_1RT}{g}, \tag{10·78}$$

where c_1 is the velocity of sound in sea water, and p is some integer usually unknown. In order to fix l, then, it is generally necessary to decrease g until coincidence is again obtained. We then have, if the new value of g is g',

$$2l = \frac{(p-1)c_1RT}{g'},$$

whence

$$p = \frac{g}{g - g'},$$

and

$$l = \frac{c_1 RT}{2(g - g')}. \qquad (10{\cdot}79)$$

From the equation it is clear that if accurate results are to be obtained the period of rotation of the horizontal wheel must be maintained very constant. In practice this wheel is run by a rotary converter whose speed is controlled by an electrically driven tuning fork. It is stated that variations in the voltage supply to the converter not exceeding 10% are smoothed out, and constancy of rotation obtained by fairly simple adjustment. It should also be mentioned that it is necessary to have the intensity the same at the two ears in order to make the comparison accurate.

It is worth noting here that Langevin has applied the use of the high frequency sound to the problem of depth finding. The details of the method do not differ much from the description of his general high frequency apparatus given in Section 10·9. Reference should be made to F. Collin, Le Génie Civil, **86**, 38–41, 1925; also **86**, 64, 1925.

We ought to note the more recent use of the depth finder to determine the presence of ice bergs.

Incidentally we may note that another method of iceberg detection may be developed from some recent observations of H. T. Barnes[1] that loud deep characteristic sounds can be heard in the underwater microphones up to three miles distant from a berg. These noises are believed to be due to the cracking of the berg under water. In the observations noted the succession of cracks occurred at the rate of from 11 to 68 per minute.

[1] H. T. Barnes, Nature, **124**, 337, 1929 (Aug. 31).

CHAPTER XI

Architectural Acoustics

11·1. Historical Introduction.—Although the acoustics of closed spaces has been of obvious importance ever since people began to gather in auditoriums, it was curiously enough not until 1895 that the maze of superstition concerning it began to be replaced by a scientific study. Architectural acoustics may be said to have had its beginning in the work of W. C. Sabine, who carried out in Cambridge, Mass., the first important experiments on the acoustical properties of a room. Up to that time the common method for "improving" the acoustical properties of an auditorium in which the hearing was pronounced bad had been to string wires. Sabine's work soon led him to an appreciation of the real problem involved, namely, the relation of reverberation to the amount of sound-absorbing material present. This matter will be considered in the following section.

11·2. Reverberation.—It has already been pointed out in Section 1·2 that by *reverberation* we mean the effect produced when a multitude of echoes follow each other in rapid succession, and in spite of the sound interference a greater sound intensity is built up than would be the case if the same source were acting in the open air and far from the presence of reflecting surfaces. If a continuous source operates in a closed space, it is seen that only the absorption by the surrounding surfaces prevents the intensity from becoming indefinitely great. Hence the magnitude of the reverberation will be controlled by the absorbing power of the surfaces. The shape of the room is in general not important except in so far as it enters into the volume. This does not mean that the presence of very deep recesses in a room of given volume will not alter the reverberation from the amount characteristic of a room of the same volume without the recesses. It does mean that, leaving recesses out of consideration, in general the curvature of the walls and their relative orientation will have negligible influence on the reverberation. Hence in the approximate theory to follow no attention is paid to the relative magnitudes of room dimensions.

We shall now derive the fundamental differential equation for the growth of sound intensity in a room. The method we shall use [1] seeks to determine the rate at which sound energy strikes the walls of a room. Consider the figure (Fig. 11·1) in which dS is an element of absorbing surface with normal NO. At a distance r from dS, where r makes the angle θ with the normal, there is an element of volume dV. Let the average density of the sound energy be uniform throughout the space under consideration and be denoted by E. The energy at any instant included in dV is then EdV and that fraction of it which will ultimately strike dS is included in the solid angle

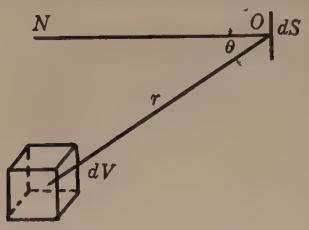

FIG. 11·1.

$$d\omega = \frac{dS \cos \theta}{r^2}.$$

Since this fraction is $d\omega/4\pi$, it follows that the amount of energy in dV which will strike dS is

$$\frac{EdVdS \cos \theta}{4\pi r^2}. \tag{11·1}$$

If the distance r is equal to c, the velocity of sound, sound energy will reach dS within one second from all the volume elements within the hemisphere of radius c with dS as center. Hence we are led to take as our volume element the zone of a hemispherical shell bounded by r and $r + dr$, θ and $\theta + d\theta$. That is,

$$dV = 2\pi r \sin \theta \cdot r d\theta dr.$$

Therefore, substituting and integrating, we have, for the rate at which sound energy falls on dS from all possible directions,

$$\frac{EdS}{2} \int_0^c \int_0^{\pi/2} \sin \theta \cos \theta d\theta dr = \tfrac{1}{4} EcdS, \tag{11·2}$$

or the rate at which sound energy falls on a unit area is

$$\tfrac{1}{4} Ec. \tag{11·3}$$

[1] See, for example, E. Buckingham, Theory and Interpretation of Experiments on the Transmission of Sound through Partition Walls, Bur. of Standards Sci. Papers No. 506. See also E. A. Eckhardt, Jour. Frank. Inst., June 1923.

Now let the absorption coefficient or power of the element dS be denoted by α. Then the absorbing power of the whole surface concerned (which is the fraction of the incident sound energy which is not thrown back into the room) is

$$a = \int_0^S \alpha dS = \bar{\alpha} S, \tag{11·4}$$

where $\bar{\alpha}$ may be defined as the *average* absorbing power per unit area of the exposed surface. The rate at which energy is absorbed by the surfaces of the room is therefore

$$\tfrac{1}{4} E c \bar{\alpha} S.$$

Now the rate at which the sound energy is absorbed in the walls of the room plus the rate at which it increases throughout the room must equal the rate at which it is being produced. We have therefore for our fundamental differential equation

$$V \frac{dE}{dt} + \tfrac{1}{4} E c \bar{\alpha} S = A, \tag{11·5}$$

where V is the volume of the room and A is the rate of sound production, assumed constant. The solution of this equation, assuming production to start at $t = 0$, is

$$E = \frac{4A}{c\bar{\alpha}S}(1 - e^{-(c\bar{\alpha}S/4V)t}), \tag{11·6}$$

whence after the steady state has been established we have for the maximum density

$$E_{\max} = \frac{4A}{c\bar{\alpha}S}. \tag{11·7}$$

If then the source is stopped, the decay will follow eq. (11·5) with $A = 0$. That is,

$$E = E_{\max} e^{-(c\bar{\alpha}S/4V)t}. \tag{11·8}$$

This equation was first derived by W. S. Franklin [1] in 1903. The time it takes the sound energy density to decay to $1/e$th of its maximum is thus $T_e = 4V/c\bar{\alpha}S$. But this quantity is too small to be of much practical use. Sabine decided to start with the minimum audible energy density and to use 10^6 times this minimum as the standard saturation energy density at which decay begins. He

[1] W. S. Franklin, Phys. Rev, 16, 372, 1903.

then defines as the *reverberation time* the time in which the decay takes place from one value to the other. That is the time given by

$$10^{-6} = e^{-(c\bar{\alpha}S/4V)t}$$

or

$$T = .161\frac{V}{\bar{\alpha}S} = .161\frac{V}{a} \qquad (11\cdot9)$$

if all linear dimensions are in meters and we use $c = 344$ meters/sec., the velocity of sound at room temperature, i.e., 20° C. If everything is expressed in terms of feet (i.e., V cu. feet, S square feet and c in feet/sec.), we have the form

$$T = .049\frac{V}{\bar{\alpha}S} = .049\frac{V}{a}. \qquad (11\cdot10)$$

The reverberation time T can then be used as a measure of the rate of intensity decay of sound in a room. It is well to point out here the distinction between energy density and intensity. The former is the amount of energy per unit volume, while the latter is the rate of flow of energy per unit area. As has been shown in Section 1·15, for a plane wave the latter equals the former multiplied by the velocity of sound. Hence the ratio of two values of the energy density is the same as that of the corresponding values of the intensity. It is therefore customary in using the reverberation time to refer to the decay of intensity instead of energy density.

We have derived the expression for the reverberation time by mathematical analysis which neglects the dissipation of sound energy by viscosity and heat conduction and which also disregards entirely interference effects. The interesting fact is that the purely experimental work of Sabine [1] gave for the reverberation time

$$T = .164\frac{V}{a} \qquad (11\cdot11)$$

for most of the auditoriums which he tested. The agreement between experimental and theoretical results shows that the fundamental assumptions on which the theory is based are reasonable. In particular the assumption that we need not consider the *shape* of the room in the calculations appears to be justified in auditoriums generally. Of course relatively deep recesses do not occur in the auditoriums tested.

[1] W. C. Sabine, Collected Papers on Acoustics.

It is therefore clear that to calculate the reverberation time we need know only the volume of the room and its absorbing power, and since the latter can be changed by the insertion or removal of absorbing materials, the magnitude of T is subject to control. In computing the absorbing power it is customary to group all areas having the same absorption coefficient and multiply each one by its appropriate coefficient, summing up finally for the whole. That is, in practice

$$a = \sum_{i=1}^{i=n} \alpha_i S_i \qquad (11{\cdot}12)$$

instead of the integral (11·4). This equation implies that the particular arrangement of the absorbing surfaces has no effect on the total absorbing power, an implication apparently justified by the facts. The important matter of the measurement of the absorbing power of rooms is treated in the next section.

Recent investigations by M. J. O. Strutt [1] have shown that the law of Sabine expressed in eqs. (11·9) and (11·11) is of greater generality than has been hitherto suspected. In particular the usual demonstrations of the law (including the one given above in this section) are incomplete since they neglect phase relations. Strutt shows that Sabine's law is a general asymptotic property of the forced oscillations of a continuous medium with arbitrarily distributed absorption, as the ratio of the forced frequency to the lowest characteristic frequency of the oscillating system tends to infinity. It is assumed that the absorption is a *bounded* function of the frequency (a condition generally attained in architectural acoustics). Another interesting result obtained by Strutt is that, when a system with absorption is excited by an oscillatory source of high frequency, the rate of increase follows the same law as the rate of decay in experiments like those of Sabine. He also concludes that the increase of the oscillations of the system is *complementary* to the decay, that is, the *sum* of the amplitudes taken at the same time t, reckoned in the one case from the moment the source *stops* and in the other case from the moment the source *starts*, is constant, i.e., independent of the time t. Oscillograms of the increase and decay of sound in a room, taken with great care to keep conditions uniform, confirm this conclusion.

Strutt also emphasizes the important fact that the decay of sound intensity in a room is not simply exponential in time but is

[1] M. J. O. Strutt, Phil. Mag., 8, 236, 1929.

oscillatory in nature. The experimental oscillograms actually indicate the presence of floating interference patterns.

More recently Schuster and Waetzmann [1] and Eyring [2] have pointed out that Sabine's formula (11·10) applies essentially to "live" rooms, i.e., rooms with small absorption and large reverberation. Recent experiments show that this formula does not work well for "dead" rooms, i.e., rooms where the average absorption coefficient $\bar{\alpha}$ is greater than 0.5. Eyring has derived a new formula by considering the reflection at the walls as due to a sequence of *image* sources which all come into action the instant the source starts. Sound energy reaches any given place from each image in turn. Decay begins with the simultaneous stopping of source and all the images, and the effect at each place dies out as the sound from the various images dies off. If T is defined as the reverberation time in the usual manner, on the above view the effect of *all* the image sources is 10^6 times greater than the effect of all those located *beyond* a distance cT, where c is the velocity of sound. Eyring neglects interference effects. On his theory the general decay equation takes the form

$$E = E_{\max} e^{cS \log (1-\bar{\alpha})t/4V}. \qquad (11\cdot 8a)$$

The quantities have the same significance as in (11·8). It is noted that, when $\bar{\alpha} < < 1$, (11·8a) reduces precisely to (11·8). The more general formula (11·8a) leads to the following expression for the reverberation time:

$$T = \frac{.05V}{-S \log (1 - \bar{\alpha})}, \qquad (11\cdot 10a)$$

agreeing with (11·10) if $\bar{\alpha} < < 1$. However, it differs very widely from the latter if $\bar{\alpha} > .5$. The factor 0.05 results from the assumption of perfectly *diffused* sound energy. In cases where there is an *ordered* distribution, this should be replaced by K and given a value appropriate to the distribution. Experimental tests indicate that (11·10a) is a much better representation of the reverberation time for "dead" rooms than Sabine's original formula (11·10). The practical importance of the new formula lies in the fact that it indicates that much less absorbing material is needed to make a room of specified (small) reverberation than would be calculated by the old formula.

[1] Schuster and Waetzmann, Ann. d. Phys., 5, 1, 671, 1929.

[2] Carl F. Eyring, Jour. Acous. Soc. Amer., 1, 217, 1930.

11·3. Methods of Absorption Measurement.—The measurement of the absorption of sound in a room is clearly one of the greatest importance in the practical applications of architectural acoustics. Reference to the reverberation theory of Section 11·2 indicates that several methods are available. In the first place consider eq. (11·7). This allows us to express the total absorbing power of the objects in a room in terms of the maximum steady state energy density, that is, in the form

$$a = \frac{4A}{cE_{\max}} \cdot \qquad (11\cdot13)$$

As a second possibility we may note the equation for the decay of sound after the source has been stopped, i.e., eq. (11·8). This suggests the use of Sabine's reverberation time eq. (11·9), and indeed this method is the one which has been most generally employed for the determination of a. The simplicity of the method whereby an observer stops a tone of intensity about 10^6 times the minimum audible intensity and then measures the duration of audibility has doubtless appealed to investigators.

In the actual carrying out of the reverberation method by Sabine and others [1] two sources were used, of emission rates A_1 and A_2, respectively. If the durations of audibility with the two sources are t_1 and t_2 respectively, and I_1 and I_2 are the steady state or maximum intensities (these are respectively proportional to the densities E_1 and E_2, as we have indicated above), and if the minimum audible intensity is denoted by I_m, we have from (11·8)

$$I_m/I_1 = e^{-(ca/4V)t_1}, \qquad I_m/I_2 = e^{-(ca/4V)t_2},$$

whence

$$a = \log A_2/A_1 \cdot 4V/c(t_2 - t_1), \qquad (11\cdot14)$$

since in the steady state $I_1/I_2 = A_1/A_2$ by eq. (11·7). The only measurements necessary besides V, the volume of the room, are t_2 and t_1, which can be determined by ear with the assistance of a chronograph. The simplicity of the method is evident, but Knudsen [2] points out some serious drawbacks. These are: (1) the difficulty in ascertaining exactly the time when minimum audibility is reached, due partly to the low sensibility of the ear to small changes in intensity near the threshold of audibility and partly to

[1] See W. C. Sabine, Collected Papers, Harvard University Press, 1922. The present section is based in the main on the excellent summary by V. O. Knudsen, Phil. Mag., Series 7, 5, 1240, 1928.

[2] V. O. Knudsen, loc. cit., p. 1246.

the actual fluctuation in intensity in the room produced by the sound interference pattern. The second of these influences may be minimized by the rotation of a large reflecting surface in the room. (2) The necessity for absolute quiet, since the masking effect of a residual noise on a tone near minimum audibility is very pronounced. (3) The human factor introduces uncertainty into the results of this method. It should be the effort of experiment to reduce this to a minimum. (4) In addition to the slowness with which the measurements must be carried out, must be cited the fact that the method is very inaccurate when applied to rooms in which the reverberation time is short, say of the order of 1·5 seconds or less.

The method of measuring a which is based on the eq. (11·13), i.e., that which Knudsen calls the intensity method, appears to be much more direct. Suppose first the energy density E_{max} is measured by means of a suitable detector, e.g., a telephone receiver in a room with total absorbing power a. Then let there be added a *known* amount of absorption, e.g., a number of windows opened, which may be denoted by a'. Corresponding to this, a new maximum steady state density E_{max}' will result for the same sound production A, so that we have the two equations:

$$a = \frac{4A}{cE_{max}}, \qquad a + a' = \frac{4A}{cE_{max}'}.$$

Then

$$a = \frac{a'}{E_{max}/E_{max}' - 1}. \tag{11·15}$$

This method has several advantages, namely, the facts that all the measurements are instrumental and thus independent of the human ear, that the percentage error in a is the same as the percentage error in measuring E_{max} (i.e., $\Delta a/a = -\Delta E_{max}/E_{max}$ from eq. (11·13)), and that the disturbing effect of residual noise in a room can be rendered negligible. Yet there are disadvantages, viz., reflection at the walls produces an interference pattern (already mentioned above in connection with the reverberation method), and the rate of emission of sound is influenced by this pattern. This difficulty may be overcome to a certain extent by "mixing" the sound in the room by the motion of reflecting vanes or by the motion of a source which generates sound of continuously changing frequency between certain limits. The detecting devices can also be put in motion. In the paper mentioned Knudsen has devised a simple and accurate plan

for using this method. The results give a to within 2–3% of the values obtained by the most careful application of the reverberation method. It appears that the intensity method is susceptible of great development. The plan, also mentioned by Knudsen, of taking oscillograms of the decaying sound in a room, while interesting, does not seem practical as yet.

11·4. Optimum Reverberation Time.—Applications.—It is at once evident from the equations of sound growth and decay (11·6) and (11·7) that the reverberation time is a very important factor in the use of an auditorium for hearing speech or music. For if the reverberation time is relatively great, any given sound like a spoken syllable or musical note will take a relatively long time to build up and decay and may seriously overlap any succeeding sound. Thus speech may be rendered inarticulate and music may be hopelessly blurred. On the other hand a relatively short reverberation time implies a great gain in the distinctness with which the separate sounds are heard. This matter is very well illustrated by the following graphs taken from Eckhardt.[1] The first (Fig. 11·2) shows the growth and decay of sound intensity in a room in which the reverberation time is long. The sounds in this case are supposed to be spoken syllables with an average emission time of 0.2 second and an interval of .05 second separating syllables, in which interval emission is supposed to cease. The intensity-time curves for the individual syllables are drawn at the bottom of the graph. The dotted curve is the integrated intensity or rather energy density to

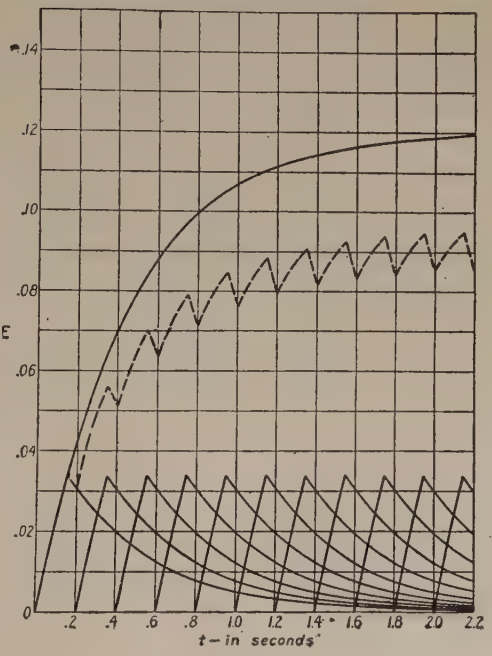

FIG. 11·2.

[1] E. A. Eckhardt, loc. cit., p. 805 ff.

which it is proportional, and the upper full line curve shows the intensity that would be produced by *continuous* emission at the same rate. It is very evident that comfortable hearing in such a room would be impossible, for the intensity builds up rapidly with only a very small fluctuation to mark the individual syllables. Only very slow speaking could be clearly understood in such a room.

Figure 11·3 shows the effect of increasing the absorbing power and hence cutting down the reverberation time. The intensity no longer builds up and there is an adequate fluctuation to render

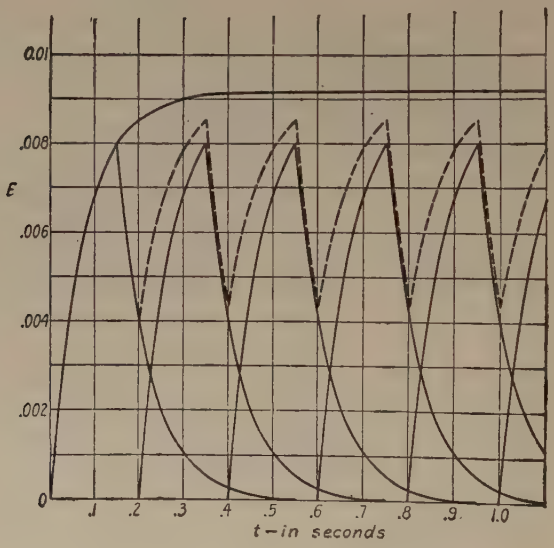

Fig. 11·3.

distinct the individual syllables. An extreme case is presented in Fig. 11·4 in which there is practically no overlapping whatever of the individual sounds. The usual observer would call such a room "dead," probably because the maximum intensity attained is so small and because most people are accustomed to a moderate amount of reverberation.

The problem of the proper reverberation time for a room is thus an extremely important one. It was first attacked by W. C. Sabine,[1] who changed the absorbing powers of various rooms by the introduction of varying amounts of absorbing material and then had

[1] W. C. Sabine, Collected Papers on Acoustics, p. 71.

musical experts listen to piano music in these rooms and pass judgment on the effect. The experts agreed remarkably well among themselves and the consensus of opinion was that 1.1 seconds is the optimum reverberation time for piano rooms. It is obvious that this method reduces the determination of the optimum time to a matter of taste, within limits, of course, and even though the tastes

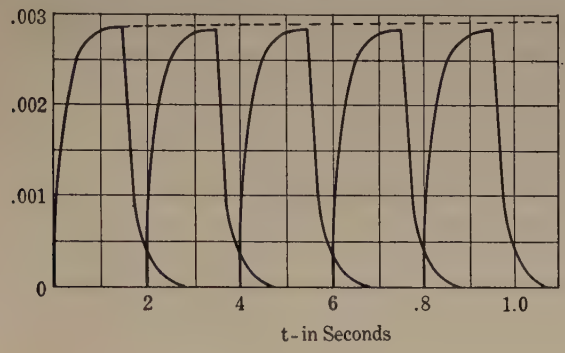

FIG. 11·4.

of musical experts may agree with regard to small piano rooms, it is not inconceivable that they may differ much with regard to large auditoriums for concert use. Moreover, it is probable that the optimum time is something which for any one observer can be altered within limits by experience. It is known, at any rate, that band masters and orchestral leaders can gradually grow so accustomed to new conditions, which were at first very distasteful, as actually to prefer ultimately the new to the old. With regard to music, therefore, it seems unwise to set a definite optimum time of reverberation. It is understood, of course, that the presence of the audience in the auditorium adds to the absorbing power and thereby renders the reverberation time a function of another variable. Reference should here be made to the work of Watson,[1] who has sought to establish a more or less definite connection between the "optimum" reverberation time of an auditorium and its volume. Moreover, he has pointed out the advantage of having the non-absorbent material in an auditorium placed in the neighborhood of the source of the sound and the absorbent material concentrated near the listeners.

[1] F. R. Watson, Jour. of Frank. Inst., July 1924. See also Science, 54, 209, 1926.

With regard to speech the matter stands on a somewhat different footing. For speech is either understandable or not and this will obviously depend on the reverberation time in a way which can be standardized. Knudsen [1] has performed some interesting experiments based on the method of speech articulation tests. These were introduced by telephone engineers in testing the speech transmission efficiency of telephone equipment. The method is somewhat similar to the old-fashioned "hearing" tests, in that meaningless monosyllabic speech sounds are called out at a definite rate in one part of a room and listeners stationed in various parts of the room record what they hear. The percentage of speech sounds heard correctly is called the percentage syllable articulation under the conditions of the test. Similar tests are performed using vowel and consonant articulation. According to Knudsen the average auditor can not understand speech in an auditorium in which the syllable articulation is less than 65%. From 65% to 85%, conditions are acceptable to the attentive listener. Above 85% the hearing is perfectly satisfactory.

The results of experiments performed in a small room ($V = 4096$ cu ft), with reverberation time controlled by the presence of absorbing material, are summarized in the following figure [2] (Fig. 11·5)

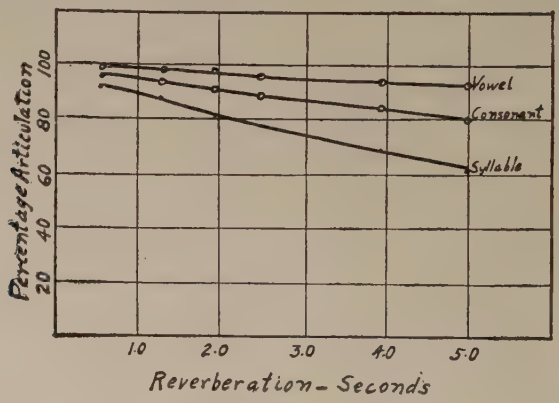

FIG. 11·5.

in which the percentage articulation is plotted against reverberation time. We clearly see that decreasing the reverberation increases the

[1] V. O. Knudsen, Phys. Rev., 26, 133, 1925; also, The Architect and Engineer, Sept. 1926, Jan, 1927.

[2] Taken from V. O. Knudsen, l.c., as are also Figs. 11·6 and 11·7.

articulation. The work was carried down to $T = 0.6$ second, but there is reason to believe that further decrease in T would lead to further increase in articulation. As a matter of fact experiments performed in the open with reverberation practically zero gave better results than the small room with $T = 0.6$ second.

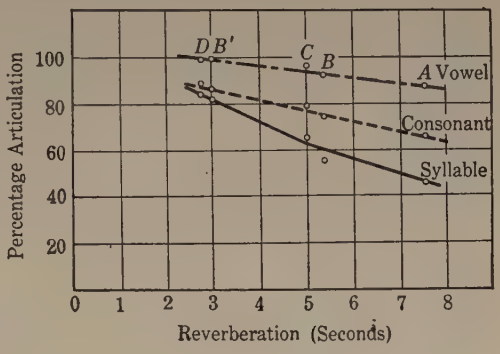

Fig. 11·6.

The next figure (Fig. 11·6) shows the similar results obtained in five high school auditoriums (without audience present) with volumes of approximately 300,000 cu ft. The indication is that the maximum reverberation time for auditoriums of this size for really good hearing conditions is about 2.75 seconds. Further experiments on this point are in progress.

Knudsen [1] has also made some interesting studies of the influence of interfering noises on the percentage articulation. The results are seen from the accompanying graph (Fig. 11·7) in which the articu-

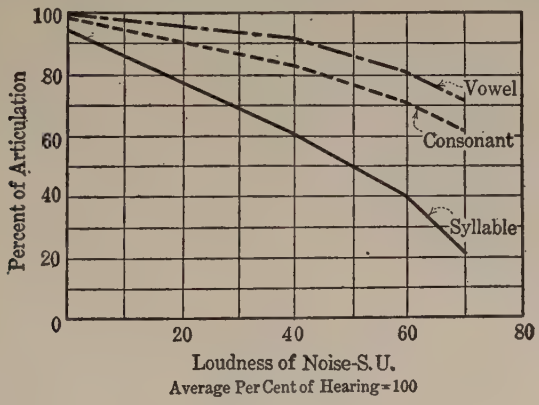

Fig. 11·7.

lation is plotted against the loudness in sensation units (S. U.).[2] When it is considered that the loudness of the ordinary conver-

[1] V. O. Knudsen, loc. cit.

[2] This unit of loudness is defined in Section 9·7.

sational voice in an auditorium is about 47 S. U., it becomes clear that disturbing noises can affect articulation to a serious extent. It may be concluded that the speech must be of the order of 30–40 S. U. louder than the noise if the latter is not to produce harmful interference. Experiments with pure tones show that the latter produce an interfering effect which for tones with loudness less than the loudness of the speech is almost independent of pitch, but which varies inversely with pitch for tones louder than the speech. In general, noise interferes more than an equally loud tone of any pitch.

11·5. Absorbing Materials.—In the previous sections attention has been called to the importance of the presence of sound-absorbing material in a room. But we have so far not considered the nature of this material. In the first place we must define a unit of absorption. An open window reflects negligible sound and hence is an almost perfect absorber. We can therefore take one square foot of open window space as the unit of sound absorption. In the metric system one square meter may be used, but it is customary in this country to use the English units in architectural acoustics. The absorption coefficient of a given kind of material is then defined as the number of absorption units per square foot, i.e., the ratio between the absorbing power of a given area of the substance and that of an equal area of open window. For example, if the absorption coefficient of a given kind of hair felt is 0.5, it means that a given area of this hair felt absorbs half the sound that would be absorbed by the same area of open window. The absorption coefficients of substances can be measured by bringing into a room known surface areas of the substance being tested and then determining the change in absorbing power by one of the methods described in Section 11·3. In this way tables of absorption coefficients have been drawn up, one of which will be found in Appendix VII (based on the work of P. E. Sabine). It must be remarked, however, that there is still considerable uncertainty in the precise value to be attached to any one material. The exact state of the absorbing surface, i.e., whether painted or unpainted, etc., must be indicated. Moreover, with absorbing materials for which the absorption depends primarily on the viscous resistance encountered in small channels or interstices, we must expect that the absorption will increase with the frequency of the sound, so that high notes will suffer a relatively greater absorption than low notes. Even when the state of the material and other conditions are given as precisely

as possible, different observers obtain somewhat different coefficients.[1] Nevertheless the great utility of high absorbing materials in the correction of auditoriums is proving a great stimulus to exact research in this field.[2] It may also be noted that experiments by P. E. Sabine have indicated that the absorbing power per unit area varies with the actual area involved. Thus for areas from 1 square meter to 10 square meters it decreases, but as the area increases the absorbing power becomes fairly uniform.

In connection with the measurement of the absorption coefficients of absorbing materials we ought to recall the work of Wente and Bedell already mentioned in Section 8·3 in the discussion of the measurement of acoustic impedance.[3] They have also determined the absorption coefficients of the materials put at the end of a tube. We may remark, however, that these measurements are not made under conditions actually realized in practice. For in rooms the absorption of sound takes place with the angle of incidence assuming random values, while in the above experiment the incidence is always normal to the absorbing surface.

11·6. Acoustic Adjustment of Rooms.—It is hard to overemphasize the great importance of adequate attention to the acoustic properties of a room or auditorium during the process of designing it and before the actual construction. The main purpose of the hall should be carefully considered and an approximate value of the desired reverberation time agreed upon. The actual absorbing power for a hall of the desired size should then be calculated from a knowledge of the area and absorption coefficients of the various absorbing materials that will enter into it, viz., the walls, ceiling, floor, seats and audience. The last will naturally be a variable factor, but in general it is sufficient to allow for an absorption corresponding to from $\frac{1}{3}$ to $\frac{1}{2}$ the maximum audience.[4] After all these factors have been accounted for, the residual necessary absorption must be introduced in a manner best suited to the architectural plan of the room. There are certain common materials for this purpose such as hair felt, plaster, cork board and various fibers. The greatly increased interest in architectural

[1] Private communication from Dr. P. E. Sabine.

[2] A good summary is to be found in "The Absorption of Sound by Materials," F. R. Watson, Bulletin 172, Eng. Exp. Station, University of Illinois.

[3] E. C. Wente and E. H. Bedell, Bell System Technical Journal, 8, 1, 1928.

[4] See F. R. Watson, Acoustics of Auditoriums, Jour. Frank. Inst., July, 1924, p. 73.

acoustics has led to the development of many varieties of patented sound-absorbing material, references to which may be found in current literature (see Appendix VII). If the total absorbing power of the necessary room equipment proves to be too great, it is possible to reduce it by painting, varnishing or otherwise treating some of the surfaces.

In the case of a hall already constructed, the acoustic properties of which are found to need correction, the above sufficiently indicates the general procedure. The employment of the services of an experienced acoustical engineer is generally an economy.[1]

11·7. Transmission through Walls and Floors.—Sound Proofing.—The transmission of sound through the partitions of a building is often a matter of great importance and has been the subject of extensive investigation. It is not our purpose here to enter into details[2] but rather to point out a few fundamental facts. When sound waves strike a wall, a portion of the energy is reflected into the room and we say that the rest is absorbed. Of the "absorbed" energy, a part suffers viscous damping in the pores of the wall and is dissipated into heat. If there are cracks or holes, a part of the energy will travel through these as sound waves in air. Another part of the energy will set up true waves in the wall which will be transmitted through it as through any medium. But it must be emphasized that, because the wall is in general very much more dense than the air, the amount of sound energy conveyed through the wall in this way is negligibly small. Lastly, the incident sound waves on the wall will cause the wall to vibrate as a whole, i.e., as a diaphragm, and so absorb energy from the incident sound and generate sound in the air on the other side. It is in this way that most of the sound energy is transmitted from the air of one room to that of the next. The actual determination of the amount transmitted in a given case is a problem of some difficulty, for a discussion of which reference may be made to Buckingham. But it seems clear that to reduce the transmission in this way the walls should be made as massive and rigid as possible and that the use of real sound-absorbing material in the construction is of little value in comparison.

Sabine[3] has demonstrated that for masonry partitions ranging in

[1] For further practical details the reader may consult F. R. Watson, Acoustics of Buildings, 1923. Second Ed., 1930.

[2] See the article of Buckingham above referred to for a good summary.

[3] P. E. Sabine, Phys. Rev., 27, p. 116 (1927).

weight from 10 to 45 pounds per square foot the reduction in sound intensity, or the ratio of incident to transmitted intensity, is proportional to the five-halves power of the weight in pounds per square foot. Another interesting experiment by Sabine [1] showed that the drumskin action of a wall is clearly in evidence in a double-walled partition. Two walls, each of single gypsum tile, were separated by an air space of two inches. The effect of bridging and filling in the space with wood was tried. Then the ratio of intensities on the two sides of the double-walled partition was measured. It was greatest for no filling at all, and the other conditions, stated in order of decreasing reduction of intensity were: felt-filled, slag-filled, bridged with wood, sawdust-filled. The reduction for the empty space was four times that for the sawdust-filled space and eighty times that for a single wall.

Investigations of essentially similar nature have been carried out by Davis and Littler,[2] whose results parallel those of Sabine to a large extent. In particular they emphasize the characteristic drumskin action. Of course it should be emphasized that the transmission about which we are here speaking is from air to air through the wall. The transmission of sound which originates *in* the wall or floor is naturally a different matter. In this case there may be considerable wave transmission through the partitions and the more rigid the construction the greater the transmission. Hence it arises that a room may be proof against sounds which arise in the air of the adjoining room (such as conversation or music), while it is not proof against sounds which arise by pounding on the floor or walls or the throbbing of machinery fixed rigidly to the floor. These facts must be considered in the practical application of sound proofing.[3]

Absorbing materials placed in wooden floors are somewhat disappointing because the drumskin action occurs. Nevertheless in home building if precautions are taken to prevent the top floor from having solid contact with the underfloor, through nails or

[1] P. E. Sabine, loc. cit.

[2] Phil. Mag., 7, 1050, 1929.

[3] Because of the nature of drumskin action, one would expect a distinct change in the transmission with frequency of a stretched membrane of appreciable mass. The higher the frequency, the less the relative response, except near resonance. Thus a piece of stretched burlap heavily painted will transmit low frequencies as a drumskin, but will more effectively reflect high frequencies. It is thus possible to make a studio selectively reverberant.

otherwise, and if the top flooring rests upon a half inch or more of absorbent material, fairly good results are obtainable.

11·8. Impulsive Sounds.—So far in our study of architectural acoustics we have been considering in general the effects of *sustained tones* only. In particular, in the derivation of the reverberation equation this was the type of sound assumed. Yet in the discussion of the influence of reverberation on the articulation of speech (Sec. 11·4) it was seen that this influence varies with the nature of the speech, that is, whether vowel, consonant or syllable. It was found, in fact, that the percentage articulation for a given reverberation time is least for syllabic sounds and greatest for vowel sounds with the consonantal sounds occupying a median position. Now a pure vowel sound has more of the characteristics of a sustained tone than the syllabic sound, which is more nearly impulsive, in spite of the fact that it contains a vowel sound. Hence the interesting question arises whether we have a right to apply to impulsive sounds without modification the theory that we have applied to sustained tones. It would seem after a moment's consideration that we have no right to do this. In fact, any one can carry out a simple experiment, showing that the reverberation time of a room, as computed from the decay of a hand clap (i.e., an impulsive sound), is considerably greater than that computed by the Sabine or other methods using sustained sounds. This is what we should expect when we remember that in the usual determination of the reverberation time the sound intensity is already at a maximum (steady state condition) when the source is stopped and the decay sets in, while in the case of the impulsive sound the intensity rises from zero to a maximum and then decreases.

As a practical example of the importance of impulsive sounds we may consider the reduction of office noises, i.e., typewriting and the like. This problem has been investigated by P. E. Sabine,[1] who has made the necessary modification in the original reverberation theory. This is summarized in the following equation:

$$E/V = e^{cat/4V},$$

where E is the total sound energy produced by a single impact, V is the volume of the room, a the total absorbing power of the room and t the duration of audibility of the impulsive sound. In the above equation, c (the velocity of sound) and V are supposed known and t

[1] P. E. Sabine, The American Architect, May 24, June 7, June 21, 1922.

can be measured in the usual way. Thus E is given in terms of a, and if one can be measured the other is known. By using two values of E, say E_1 and E_2, in successive impacts and measuring the corresponding time intervals t_1 and t_2, a is at once obtained. It is assumed that the ratio E_1/E_2 is the same as the ratio of the mechanical energies of impact. For the results of this method as applied to the problem of office noises the reader is referred to Sabine's articles.

11·9. Machinery Noises and Their Prevention.—All machinery motion involves the production of both sustained and impulsive sounds. The elimination of such sound and noise is not particularly important from the viewpoint of mechanical efficiency, for only a relatively small amount of energy is involved. Yet the disturbing effect on articulation (see Sec. 11·4) and the undesirable psychological effect which recent studies have brought to light render it desirable to reduce machinery noise as much as possible. We wish to consider in this section a few fundamental points bearing on this problem.

The general principle involved in the reduction of machinery noise is, of course, the prevention as far as possible of the transmission of vibratory energy to surfaces sufficiently large to give off appreciable acoustical energy to the air. Thus the problem is mainly one of preventing the transmission of vibration from the moving parts of the machine to the stationary foundation, for in general the amount of noise transmitted directly to the air from the moving parts is much less than that due to the vibration of the supports and foundation. For slow motion machines, where the frequency of the chief vibration is not audible, care should be chiefly devoted to the precise fitting and adequate lubrication of all gears and bearings. In the case of high speed motors and turbines the matter of support is more important. In particular any support which resonates with respect to any of the principal modes of vibration of the machine should be carefully avoided, a matter of great importance when rigid supports are necessary. The endeavor should be made, and may often prove successful, to utilize the supports to damp the vibrations taking place in them.

The support of intrinsically noisy machinery should be such as to diminish the transmission of vibration to the floor or other construction of large area. Obviously, this can be obtained by the insertion of material of very different acoustic resistance. Thus a

motor may have its bed supported on the floor by a series of springs. Since the specific acoustic resistance is $\rho_0 c$ (Sec. 3·1), it is of importance that the velocity c of a longitudinal wave in a spring is very much less than that in a continuous solid. Indeed the specific acoustic resistance of the spring for longitudinal vibrations is much less and there is considerable reflection of the longitudinal vibrations normal to the surface of contact of the motor base and the spring. Obviously, a transverse vibration could communicate but little energy also to the spring. So the effect of the springs is that of causing a reflection of vibratory energy and of seriously decreasing any vibratory motion in the floor. This provides a good illustration of the application of the principle of securing a large change in specific acoustic resistance. To summarize, it may be said that the prevention of noise from machinery can wisely be secured by the following steps: (1) prevention of the vibration at the source by reduction of the cause, (2) diminution of the conduction of the vibration to surfaces of larger area by securing large changes in specific acoustic resistance, and (3) the absorption of any persistent vibration by material in which damping is very marked.

The support of machinery on springs is sometimes discussed under the term "vibration absorbers," [1] though as a matter of fact the interest should be in the transmission of vibratory energy. The springs do not so much absorb energy as decrease the flow of energy.

But there is a method for the reduction of vibrations in machinery to which reference should be made. If a small vibratory system, tuned to the operating frequency of the machine, is attached to that machine in a suitable location, the forces set up will diminish the original vibrations.[2] Unfortunately this additional system is called a "dynamic vibration absorber," though the term "absorption" in physics usually refers to energy. It is possible to add damping to the small system and thereby make it serve over a broader range of frequencies. Dynamic vibration "absorbers" may be constructed for longitudinal, transverse or torsional vibrations. The last is important in automobile engines.

[1] For a discussion see "Vibratory Problems in Engineering," by Timoshenko, D. Van Nostrand Company, 1928, p. 33. The authors think the explanation given by Timoshenko is in error in comparing static vibratory conditions. It is the flow of energy with which one should be concerned.

[2] See Timoshenko, loc. cit., p. 101, or the contribution by Ormondroyd and DenHartog, Trans. Amer. Soc. Mech. Eng., 1928.

CHAPTER XII

Atmospheric Acoustics

12·1. Resumé of Principal Phenomena.—The passage of sound through the atmosphere has already been touched upon in connection with some of the simple properties of sound waves treated in Chapter I. It is the purpose of the present chapter to discuss in more detailed fashion the problems that arise in this connection. In this section [1] we shall undertake a brief survey of the leading phenomena of atmospheric acoustics, reviewing in part material already presented.

We have already deduced the general formula for the velocity of a compressional wave in a fluid medium, viz., $c = \sqrt{dp/d\rho}$, and have noted (Sec. 1·6) that in the case of waves in air this reduces to the form $c = \sqrt{\gamma p/\rho}$, where p is the atmospheric pressure, ρ the density and γ the ratio of the specific heat at constant pressure to that at constant volume. From the general gas equation it follows that the velocity of sound in air is *independent* of the pressure, but is *dependent* on the temperature (eq. 1·3 of Sec. 1·6). We also note that for sound waves in the open air (not confined to narrow tubes) the velocity is independent of the frequency, except possibly at very high frequencies (see note, Sec. 1·13), and the intensity of the sound. Since the atmosphere always contains some vapor, the dependence of c thereon is a matter of some interest. Both γ and p/ρ are affected by the pressure of the vapor. According to Humphreys [2] we may write for moist air

$$\gamma = 1.40 - 0.1e/p, \tag{12·1}$$

where p is the total atmospheric pressure and e is that portion of it due to the water vapor. Moreover, for the density we have

$$\rho = \rho_0 \frac{B - .378w}{760(1 + \alpha t)}, \tag{12·2}$$

[1] We shall here of necessity be under great obligation to W. J. Humphreys, Journal of the Franklin Institute, May 1921. See also, by the same author, "Physics of the Air," 2d edition, 1928.

[2] Loc. cit., p. 587.

where the significance of the various quantities is as follows: ρ_0 is the density of dry air at 0° C. and 760 mm of Hg; B is the reduced total atmospheric pressure in mm of Hg; α and t are, respectively, the coefficient of thermal expansion of the air per degree C. at 0° C. and the temperature on the centigrade scale. The substitution of the above quantities as given in (12·1) and (12·2) into the expression for the velocity indicates that c in moist air is slightly greater than that in dry air. For example, the velocity of sound in saturated air at 20° C. and 760 mm of Hg is about 0.35% greater than that in perfectly dry air at the same temperature and pressure. The influence of fog, dust and smoke on the velocity of sound in the atmosphere, though not definitely well known, is probably negligible in most cases.

In Section 1·18 we discussed the normal reflection of sound in fluid media and in Section 4·2 the general case of reflection at any angle of incidence. In the present instance it is of interest to recall (eq. (4·13)) that when a plane wave of sound is incident normally from dry air on a fog bank or cloud in which the density is roughly 1% less than that of dry air, the ratio of the reflected intensity to the incident intensity is but approximately 1/160,000. But here we have neglected the possibility of total reflection, which can take place when a plane wave of sound is incident at a sufficiently large angle in passing from one medium to another in which the specific acoustic resistance $\rho_0 c$ is less. Thus to return to Section 4·2 and eq. (4·13) we note that total reflection will result if $\cot \theta_2/\cot \theta_1$ is imaginary, so that θ_2 no longer exists. To see under what condition this will take place, we may utilize the law of refraction to write

$$\cot \theta_2/\cot \theta_1 = c_1/c_2 \cdot \sqrt{\frac{1 - c_2^2/c_1^2 \sin^2 \theta_1}{\cos^2 \theta_1}}, \qquad (12\cdot3)$$

the subscripts 1 and 2 referring to the incident and refracting medium, respectively, as usual. The ratio will then be imaginary, provided $\sin \theta_1 > c_1/c_2$. Hence the critical angle is

$$\begin{aligned} \theta_c &= \text{arc sin } c_1/c_2 \\ &= \text{arc sin } \sqrt{\rho_{02}/\rho_{01}} \end{aligned} \qquad (12\cdot4)$$

to a sufficiently good approximation in general. In the case of the fog bank above mentioned, where $\rho_{02}/\rho_{01} = 0.99$, it develops that $\theta_c = 84°$ approximately. Hence in spite of the fact that the

reflection is small at angles near to normal incidence, a few successive total reflections may largely change the direction of the sound. Of course, the assumption of *dry* air in this example must be considerably in error in the actual case.

Natural echoes have been mentioned in Section 1·2. Humphreys[1] classifies them under the following heads: (1) the discrete single echo; (2) the discrete multiple echo (a number of successive reflections); (3) the overlapping multiple echo—reverberation; (4) the diffuse echo, due to the scattering of the sound by many small objects; (5) the harmonic echo, due to the greater scattering of an overtone than of the fundamental (note that the intensity of the scattered sound varies inversely as the fourth power of the wave length); (6) the musical echo, due to reflection from, or scattering by a series of objects spaced at uniformly increasing distances from the source. In the last named case the echo will have a definite pitch dependent on the spacing of the objects and the position of the observer.

The two principal types of sound refraction by the atmosphere, viz., convective (i.e., by the wind) and temperature, have already been noted in Sections 1·7 and 1·8. There are some interesting special cases of temperature refraction which may be elaborated on here with profit. In the case of temperature inversion (i.e., the temperature increasing upwards from the earth's surface), the bending of the wave front may result in confining a large part of the sound emitted by a source near the surface in a shallow cylindrical layer of height determined by the maximum temperature attained. This is accomplished by the ordinary reflection at the earth's surface and refraction in the inversion layer. The intensity of the sound so confined falls off roughly inversely as the distance from the source and consequently the range is greatly increased. The situation is illustrated in the following figure (taken from Humphreys, p. 597).

Fig. 12·1. Sound Confined to an Inversion Layer.

The same atmospheric conditions which produce optical mirages are also effective in producing acoustical mirages. The following

[1] Loc. cit., p. 594.

figure [1] (Fig. 12·2) indicates the situation where the temperature rapidly *decreases* upward for a short distance and shows how the original wave front ABC becomes the distorted wave front $A'B'C'$.

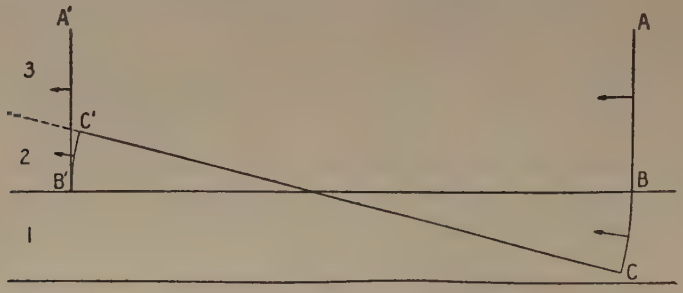

Fig. 12·2. Inferior Acoustical Mirage.

At the position 1 (in the region in which the temperature change is supposed to take place) little or no sound will be heard. At 2, two identical sounds will be heard, one from the direction of the original source and the other from a point below the latter. At 3, but one sound is heard from the direction of the original source. From the optical analogy this may be called the *inferior* acoustical mirage. What may on similar grounds be called the *superior* acoustical mirage is shown in the next figure (Fig. 12·3), wherein the tempera-

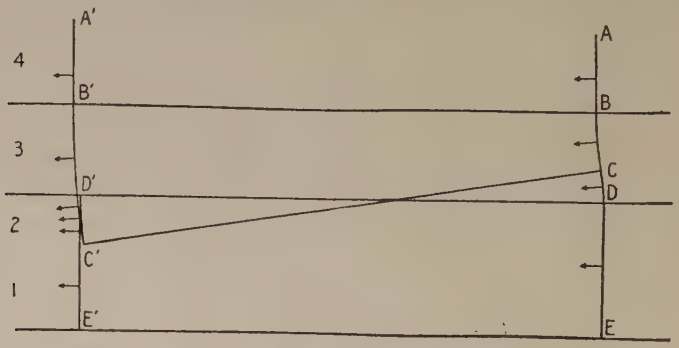

Fig. 12·3. Superior Acoustical Mirage.

ture is uniform throughout a shallow layer near the surface, then increases with elevation for several meters, finally becoming uniform at the higher temperature at a higher elevation. Then the wave front $ABCDE$ becomes the altered wave front $A'B'C'D'E'$, and at 2

[1] The figures of this section are all taken from Humphreys, loc. cit.

three identical sounds may be heard, one in the direction of the original source, one from a slightly higher direction and the third from a still higher point.

The refraction of sound due to wind currents was treated in Section 1·7. Here we need only recall the qualitative facts that, when a sound originates near the surface of the earth and there is a wind blowing with velocity increasing upwards, the wave fronts are lifted to windward and forced down to leeward, much decreasing the range in the former direction and increasing it in the latter. The following figure (Fig. 12·4) is illustrative of this point. The ad-

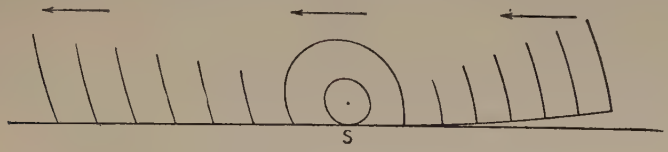

Fig. 12·4. Distortion by Wind of Sound Fronts.

vantage of high elevations for sound sources for transmitting to windward is patent. Often sound from the foot of a mountain will be carried over the top to leeward by the action of the wind in bending the wave fronts progressing forward, as is well indicated in the following figure (Fig. 12·5). Oppositely directed winds in

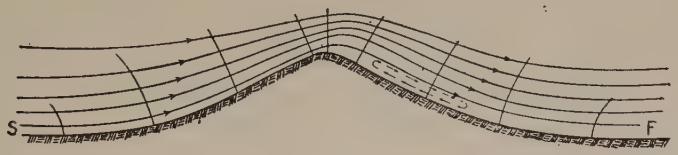

Fig. 12·5. Sound Crossing a Mountain to Leeward.

adjacent layers (a not unusual state of affairs) act to bend the sound wave fronts first up and then down, and hence may cause the zones of silence so frequently observed with auditory areas on either side.

Among the sounds of meteorological origin which the reader will find discussed in an interesting manner by Humphreys (loc. cit., p. 607 ff.), we may briefly note the following: creaking of snow, thunder, brontides, howling of the wind, humming of wires, whispering of trees, murmur of the forest, roar of the mountain, rustling of leaves, rattle of rain, etc. Though perhaps not precisely in the same class, the city dweller would doubtless like to add the hum of traffic, the

far-off puffing of the locomotive, the increasingly prominent roar of the airplane and the various noises which accompany the industry of a busy city. It may not be inappropriate to mention here the Doppler effect which must be noticed many times a day by all who are passed in the street by automobiles. Thus the motor's hum is always of higher pitch when approaching the observer than when receding from him. This effect, characteristic of all periodic wave motion, is described in detail in all texts on optics.[1]

Of all sounds such as are noted above, perhaps the most interesting are those dependent on *aeolian tones*. It has long been known that when a current of air strikes a stretched wire normal to its length, tones are produced, the pitch of which in a given case is independent of the material, length and tension of the wire, but, as was found by Strouhal,[2] is given approximately by the formula

$$\nu = 0.185v/d,$$

where v is the velocity of relative motion of wire and air and d is the diameter of the wire (all quantities in c.g.s. units). The production of these sounds is undoubtedly due to the instability of the vortex sheets which are formed by the obstruction the wire affords to the rushing air. The eddies form a mass vibrating from one side of the wire to the other. No completely satisfactory dynamical theory has as yet been given.[3] When the pitch of the tone coincides with a free oscillation frequency of the wire, these aeolian tones are rendered much louder, for then the wire itself tends to vibrate in a direction normal to the wind current. The humming of telephone and telegraph wires, the whispering of trees and the general forest murmurs are all illustrations of aeolian tones, sometimes attaining large volume when the number of active objects is large.

12·2. General Theory of the Propagation of Sound through the Atmosphere.—The fundamental problem of atmospheric acoustics may be considered to be the determination of the propagation of a sound wave through a medium in which the velocity of the wave and the velocity of the medium vary continuously from point to point. The importance of the solution of this problem for sound ranging and signalling in air is obvious. In this and the following section

[1] See, for example, Houstoun, A Treatise on Light, London, 1915, p. 275 ff. Or also Humphreys, loc. cit., p. 588.

[2] Ann. der Phys., 5, 216, 1878.

[3] But see the work of v. Kármán and Rubach, Phys. Zs., 13, 49, 1912.

we shall follow in the main the analysis of Milne [1] in deriving the general equations of propagation in a medium such as we have just indicated. The calculation is based on the following generalization of Huyghens' principle: each point of a wave front moves with a velocity which is the vector sum of (1) the wave velocity of sound at the point considered measured in the direction of the positively drawn normal to the wave front at this point, and (2) the velocity of the medium at this point. Using this principle, it is possible to follow the motion of the wave front if we have given the two velocities as functions of the coördinates. We shall then define a "ray" of sound as a curve whose tangent at each point is in the direction of the resultant velocity, taken at the moment when the wave front passes through the point.

The treatment to be given here is confined to the case of steady motion. Let the components of the medium velocity at the point (x, y, z) be u, v, w, while the velocity of sound is c, a function of x, y, z. If we denote the direction cosines of the normal to the wave front by l, m, n, the equations of the sound ray, expressed in terms of the velocity components of the point (x, y, z) on the wave front, will be

$$\begin{aligned} \dot{x} &= lc + u, \\ \dot{y} &= mc + v, \\ \dot{z} &= nc + w. \end{aligned} \qquad (12{\cdot}6)$$

It is necessary to add to these equations others expressing the variation of l, m, n along the ray. Without leaving the wave front let us vary the coördinates x, y, z by $\delta x, \delta y, \delta z$. Then, corresponding to eq. (12·6) we have the following:

$$\delta(\dot{x}) = (\dot{\delta x}) = \delta x \frac{\partial}{\partial x}(lc + u) + \delta y \frac{\partial}{\partial y}(lc + u) + \delta z \frac{\partial}{\partial z}(lc + u) \qquad (12{\cdot}7)$$

with two additional similar equations in y and z. Since the varied point lies on the same wave front,

$$l\delta x + m\delta y + n\delta z = 0, \qquad (12{\cdot}8)$$

whence on differentiation with respect to the time and substitution from eq. (12·7) for $(\dot{\delta x})$, etc., we have

$$\Sigma \delta x \left(\dot{l} + \frac{\partial c}{\partial x} + l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x} \right) = 0, \qquad (12{\cdot}9)$$

[1] E. A. Milne, Phil. Mag., 42, 96, 1921.

where the summation sign indicates the sum of three terms with factors δx, δy, δz, respectively. In obtaining the above, one needs to recall the identities $l^2 + m^2 + n^2 = 1$ and $(\partial/\partial x)(l^2 + m^2 + n^2) = 0$, etc. Now since eq. (12·9) holds for all values of δx, δy, and δz, satisfying (12·8), we must have

$$\begin{aligned} 1/l \cdot \left(\dot{l} + \frac{\partial c}{\partial x} + l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x} \right) \\ = 1/m \cdot \left(\dot{m} + \frac{\partial c}{\partial y} + l\frac{\partial u}{\partial y} + m\frac{\partial v}{\partial y} + n\frac{\partial w}{\partial y} \right) \\ = 1/n \cdot \left(\dot{n} + \frac{\partial c}{\partial z} + l\frac{\partial u}{\partial z} + m\frac{\partial v}{\partial z} + n\frac{\partial w}{\partial z} \right). \end{aligned} \tag{12·10}$$

Writing

$$V = c + lu + mv + nw,$$

where V is thus the component velocity of the point (x, y, z) along the wave normal, and differentiating V with respect to x, y and z, but considering l, m, n, as constants in so doing, so that, e.g.,

$$\frac{\partial V}{\partial x} = \frac{\partial c}{\partial x} + l\frac{\partial u}{\partial x} + m\frac{\partial v}{\partial x} + n\frac{\partial w}{\partial x},$$

eqs. (12·10) then become

$$\begin{aligned} 1/l \cdot \left(\dot{l} + \frac{\partial V}{\partial x} \right) = 1/m \cdot \left(\dot{m} + \frac{\partial V}{\partial y} \right) = 1/n \cdot \left(\dot{n} + \frac{\partial V}{\partial z} \right) \\ = \left(l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z} \right) \end{aligned} \tag{12·11}$$

by composition. Using these latter equations, it is seen that we can replace (12·10) by the three equations

$$\begin{aligned} \dot{l} &= -(m^2 + n^2)\frac{\partial V}{\partial x} + ml\frac{\partial V}{\partial y} + nl\frac{\partial V}{\partial z}, \\ \dot{m} &= ml\frac{\partial V}{\partial x} - (l^2 + n^2)\frac{\partial V}{\partial y} + nm\frac{\partial V}{\partial z}, \\ \dot{n} &= nl\frac{\partial V}{dx} + mn\frac{\partial V}{\partial y} - (l^2 + m^2)\frac{\partial V}{\partial z}. \end{aligned} \tag{12·12}$$

The three eqs. (12·6) and the three eqs. (12·12) are sufficient for a complete determination of the ray. Of these the former set give the direct propagation of the sound as affected by the bodily motion of the medium, while the latter express the refraction of the sound caused by the variation of V over the wave front.

With a view to the applications of the next section let us confine our attention to propagation through a *stratified* medium, i.e., one such that the z axis can always be chosen so that $w = 0$ and u, v and c are functions of z only. With this simplification the fundamental eqs. (12·6) and (12·12) become, if we recall that $\partial V/\partial x = \partial V/\partial y = 0$,

$$\begin{aligned} \dot{x} &= lc + u, \\ \dot{y} &= mc + v, \\ \dot{z} &= nc, \end{aligned} \tag{12·13}$$

and

$$\begin{aligned} \dot{l} &= ln\frac{\partial V}{\partial z}, \\ \dot{m} &= mn\frac{\partial V}{\partial z}, \\ \dot{n} &= (n^2 - 1)\frac{\partial V}{\partial z}. \end{aligned} \tag{12·14}$$

From the first two of eqs. (12·13) we see at once that

$$l/m = \text{constant}, \tag{12·15}$$

indicating that the projections on the xy plane of the normals to the wave fronts along any given ray have the *same* direction. In particular it is seen that the normals to the successive wave fronts along any ray will remain parallel to a fixed plane through the z axis, i.e., a plane which is perpendicular to the layers of stratification. Considering now a particular ray, let us adjust the x and y axes so that the projections of the normals on the xy plane are parallel to the x axis. This fixes $m = 0$, and hence, since $\partial V/\partial z$ reduces to $\partial c/\partial z + l(\partial u/\partial z)$, we have

$$\begin{aligned} \dot{z} &= nc, \\ \dot{l} &= nl\left(\frac{\partial c}{\partial z} + l\frac{\partial u}{\partial z}\right), \end{aligned}$$

so that

$$\frac{\partial l}{\partial z} = \frac{l}{c}\left(\frac{\partial c}{\partial z} + l\frac{\partial u}{\partial z}\right),$$

or, proceeding now to ordinary differentials,

$$-\frac{d}{dz}(c/l) = \frac{du}{dz}. \tag{12·16}$$

The integral of the above differential equation is

$$c/l - c_0/l_0 = u_0 - u,$$

where the zero subscripts refer to initial or boundary values. If we set $l = \cos\theta$, $l_0 = \cos\theta_0$, the above equation becomes

$$c \sec\theta - c_0 \sec\theta_0 = u_0 - u. \qquad (12{\cdot}17)$$

Equation (12·17) represents the general law of refraction of sound in a stratified medium when both c and u vary. Incidentally we may note (recalling that here θ is the angle which the normal to the wave front makes with the x axis) that eq. (12·17) reduces to the form

$$\sec\theta - \sec\theta_0 = \frac{u_0 - u}{c}, \qquad (12{\cdot}18)$$

when $c = c_0$, and the refraction is convective only. This is precisely the special formula (1·5) deduced directly in Section 1·7 (note that in (1·5) θ denotes the angle complementary to the angle θ of the formula above). On the other hand, if $u_0 = u$, i.e., there is no motion of the medium, we have

$$\frac{\sin(\pi/2 - \theta)}{\sin(\pi/2 - \theta_0)} = \frac{c}{c_0}, \qquad (12{\cdot}19)$$

the ordinary law of Snell (see Sec. 1·8). In the next section application is made of the equations here developed to special problems of sound ranging in air.

12·3. Application to Sound Ranging and Signalling in Air.— The problem of sound ranging in air is the detection by an observer on the ground of the true direction of the sound from an unseen aerial source such as an airplane. (For the detection of cannon location, see particularly the following section.) The procedure will, of course, involve a knowledge of the corrections which must be applied to the experimentally determined direction in taking account of the transmission of the sound through the atmosphere considered as being approximately a stratified medium. It is assumed that the instrument used measures the direction of the normal to the incident wave front. In this section we again follow Milne.[1]

[1] Loc. cit., p. 101 ff.

In Fig. 12·6[1] the observer's position is taken to be the origin of coördinates, while S with the coördinates x_s, y_s, z_s is the source of the sound. The observed direction of the source, i.e., the normal to the incident wave front at O, is OS'. Let the vertical plane through OS' be the xz plane, and let the projections of S on the xy and xz planes be, respectively, N and S_1. The observed azimuth is thus made equal to zero, while the corrected azimuth is the angle $NON' = \delta\varphi$. The observed elevation is the angle $S'ON' = E$, say, while the correction to the elevation is the angle $S'OS_1 = \delta E$. Let $u(z)$ and $v(z)$ be the x and y components of the wind velocity at height z, while $c(z)$ is the velocity of sound. By θ we shall denote the angle which the normal at any point makes with the horizontal, measured positively downwards. Now the normal at O has been taken in the xz plane. Hence by eq. (12·15) and the accompanying discussion the normal will everywhere be parallel to this plane. We can therefore write

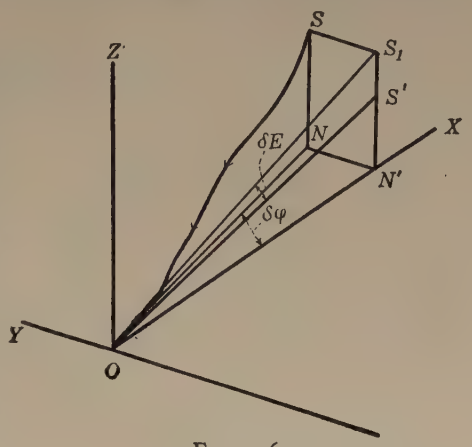

Fig. 12·6.

$$l = -\cos\theta, \qquad m = 0, \qquad n = -\sin\theta.$$

The equations of the ray connecting S and O are then (eq. (12·6))

$$\begin{aligned} \dot{x} &= -c(z)\cos\theta + u(z), \\ \dot{y} &= v(z), \\ \dot{z} &= -c(z)\sin\theta, \end{aligned} \tag{12·20}$$

while the refraction eq. (12·17) becomes

$$c(z)\sec\theta - c_0\sec\theta_0 = u_0 - u(z). \tag{12·21}$$

From the first and third equations in (12·20) and from (12·21) it is evident that the projection of the ray on the zx plane is independent of $v(z)$ (the so-called "cross wind"). Hence δE is independent of $v(z)$, and in calculating the elevation correction we can take S_1 as the

[1] Taken from Milne, with slight modification. Figs. 12·7 and 12·8 are from the same source.

source. From the second and third equations in (12·20) we have

$$\frac{dy}{dz} = -\frac{v(z)}{c(z)} \csc \theta,$$

from which it follows that

$$y_s = -\int_0^{z_s} \frac{v(z)}{c(z)} \csc \theta dz.$$

Hence

$$\tan \delta\varphi = y_s/(z_s \cot (E + \delta E))$$
$$= -\frac{\tan (E + \delta E)}{z_s} \int_0^{z_s} \frac{v(z)}{c(z)} \csc \theta dz. \quad (12\cdot22)$$

If we knew $v(z)$, $u(z)$ and $c(z)$, we could compute θ from (12·21) and hence evaluate the integral in (12·22). However, it is more convenient at this point to introduce approximations. The velocities u and v are always much smaller than c. Hence, to a rather good approximation we may set $\theta = \theta_0 = E + \delta E$ in (12·22), i.e., consider θ as a constant under the integral sign and equal to the actual elevation. Then

$$\tan \delta\varphi = -\left(\frac{\bar{v}}{c}\right) \sec \theta_0 = -\frac{\bar{v}}{c_0} \sec \theta_0, \quad (12\cdot23)$$

where $\bar{v}$ is the average $v(z)$ from O to z_s, and we have neglected the change in c as compared with that in v. The azimuth correction is then given to a first approximation by (12·23).

To get the elevation correction we proceed as follows. From the first and third of eqs. (12·20) we have

$$\frac{dx}{dz} = \cot \theta - \frac{u(z)}{c(z)} \csc \theta,$$

which yields

$$\cot (E + \delta E) = x_s/z_s = \frac{1}{z_s} \cdot \int_0^{z_s} \left[\cot \theta - \frac{u(z)}{c(z)} \csc \theta \right] dz.$$

If we subtract $\cos \theta_0$ from both sides and simplify, we get

$$\frac{\sin \delta E}{\sin (E + \delta E)} = \frac{1}{z_s} \int_0^{z_s} \frac{1}{\sin \theta} \left[\sin (\theta - \theta_0) + \frac{u(z)}{c(z)} \sin \theta_0 \right] dz. \quad (12\cdot24)$$

From the general refraction law (12·21) it develops that

$$\sin(\theta - \theta_0) = \frac{\cos\theta \cos\theta_0 \cos\frac{1}{2}(\theta - \theta_0)}{\sin\frac{1}{2}(\theta + \theta_0)} \left[\frac{u - u_0}{c_0} + \frac{c_0 - c}{c_0} \sec\theta \right].$$

Hence the integral in (12·24) can be divided into three parts, viz.:

$$\frac{\sin\delta E}{\sin(E + \delta E)} = \frac{1}{z_s} \int_0^{z_s} \frac{\cos\theta \cos\theta_0 \cos\frac{1}{2}(\theta - \theta_0)}{\sin\theta \sin\frac{1}{2}(\theta + \theta_0)} \cdot \left(\frac{u(z) - u_0}{c_0} \right) dz$$

$$+ \frac{1}{z_s} \int_0^{z_s} \frac{u(z)}{c(z)} \cdot \frac{\sin\theta_0}{\sin\theta} dz$$

$$+ \frac{1}{z_s} \int_0^{z_s} \frac{\cos\theta_0 \cos\frac{1}{2}(\theta - \theta_0)}{\sin\theta \sin\frac{1}{2}(\theta - \theta_0)} \cdot \frac{c_0 - c}{c_0} dz. \quad (12·25)$$

Again it will be found more convenient to approximate by setting $\theta = \theta_0 = E + \delta E$. The following approximate expression for the elevation correction results:

$$\delta E = \frac{\overline{u(z)} - u_0}{c_0} \cdot \cos\theta_0 \cot\theta_0 + \frac{\overline{u(z)}}{c_0} \cdot \sin\theta_0$$

$$+ \frac{c_0 - \overline{c(z)}}{c_0} \cdot \cot\theta_0. \quad (12.26)$$

In δE the three terms represent, respectively, (1) the effect of the wind in refracting the sound, (2) the effect of the wind in convecting the sound, and (3) the effect of the temperature variation. Investigating the third term in somewhat greater detail, we have, if T represents the absolute temperature (Sec. 1·6),

$$c = c_0\sqrt{T/T_0},$$

and, assuming a linear temperature gradient, viz.,

$$T = T_0(1 - 2z/C),$$

so that

$$c = c_0(1 - z/C),$$

we have finally

$$\frac{c_0 - \overline{c(z)}}{c_0} \cdot \cot\theta_0 = \frac{1}{2} z_s/C \cdot \cot\theta_0 = \frac{1}{2} x_s/C. \quad (12·27)$$

To a first approximation then the correction in elevation due to temperature variation is proportional to the horizontal distance of the source. If the temperature gradient is the not uncommon one of 1° F. per 300 ft rise, then the constant $C = 300{,}000$ ft. Investiga-

tion shows that in this case of linear temperature gradient the sound rays are arcs of circles. It must be noted that eq. (12·26) may be seriously in error for small θ_0, for in this case the correction δE may be of the order of E and θ_0 themselves. Any approximate formula will be of doubtful value in such a case.

We have already considered the occurrence of total reflection in Section (12·1). Its influence on the range of audibility will be further briefly discussed here. Reverting to eq. (12·21) we note that for certain values of z, sec θ may become *less* than unity. This is the indication of total reflection. If u and c change continuously with height, the ray will have a horizontal tangent at the point of reflection (see Fig. 12·7). Here again let S be the source and let θ

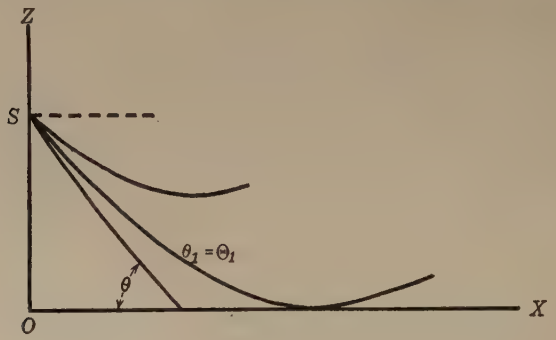

Fig. 12·7.

be the angle which the normal to the wave front makes with the horizontal. The initial and final values of θ will be denoted by θ_1 and θ_0, respectively. Then

$$c_1 \sec \theta_1 - c_0 \sec \theta_0 = u_0 - u_1, \qquad (12\cdot28)$$

where c_1 is the velocity of sound at the height of the source and u_1 is the wind velocity at the same place (we are here neglecting the cross wind). First let us set $\theta_0 = 0$, getting then for θ_1 (denoted for this case by Θ_1)

$$\operatorname{Sec} \Theta_1 = \frac{u_0 - u_1 + c_0}{c_1}. \qquad (12\cdot29)$$

In the second place, let $\theta_0 = \pi$ and get for θ_1 (denoted here by Θ_1')

$$\operatorname{Sec} \Theta_1' = \frac{u_0 - u_1 - c_0}{c_1}. \qquad (12\cdot30)$$

If $\sec \Theta_1 > 1$, the angle Θ_1 is real. Thus the condition that it should be so is

$$u_0 - u_1 + c_0 - c_1 > 0. \tag{12·31}$$

At any point of the ray for which $\theta_0 = 0$, we have

$$c \sec \theta = u_0 - u + c_0. \tag{12·32}$$

Provided then that

$$u_0 + c_0 > u + c, \tag{12·33}$$

for all heights between the source and the ground, the ray for which $\theta_1 = \Theta_1$ will actually reach the ground. Moreover, we see that all rays for which $\pi/2 \geq \theta_1 \geq \Theta_1$ will reach the ground (θ_0 real), while those for which $\Theta_1 > \theta_1 \geq 0$ will not reach the ground, since for them θ_0 is imaginary, as is evident from substitution into (12·28). Hence Θ_1 serves to fix a bounding ray, separating totally reflected rays from those which reach the ground. Beyond the point of contact of the ray $\theta_1 = \Theta_1$ with the ground the sound will be inaudible at the surface. Similarly, if at all heights

$$c_0 - u_0 > c - u,$$

the ray $\theta_1 = \Theta_1'$, where $\pi/2 < \Theta_1' < \pi$, will be a boundary ray for those in the negative x direction. Noting that as the figure has been drawn here, $l = \cos\theta$ and $n = \sin\theta$, so that

$$\frac{dx}{dz} = \cot\theta + \frac{u}{c} \cdot \cos\theta,$$

the range of audible sound on the surface is then

$$R = \int_0^{z_s} \left(\cot\theta + \frac{u}{c} \csc\theta \right) dz. \tag{12·34}$$

Now substituting from (12·32) for θ and neglecting $uc_0 + uu_0 - u^2$ as compared with c^2, and $cu_0 - cu$ as compared with $cc_0 \sim c^2$, we finally find

$$R = \int_0^{z_s} \sqrt{\frac{c/2}{c_0 - c + u_0 - u}}\, dz \tag{12·35}$$

for the range in the positive x direction. The only difference in the negative x direction is in the sign of u, which must be changed, so that for this case we have

$$R' = \int_0^{z_s} \sqrt{\frac{c/2}{c_0 - c + u_0 + u}}\, dz. \tag{12·36}$$

The integrals (12·35) and (12·36) are infinite integrals, for the denominators vanish for $z = 0$. If the integrals converge, the range of audibility is limited. If they diverge, it is unlimited and theoretically the sound is everywhere audible.

As an illustration, consider $u \pm c$ as a linear function of z. Then on evaluation the integrals yield

$$R, R' = z_s \sqrt{\frac{\bar{c}}{c - \bar{c} \pm u \mp \bar{u}}}.$$

Thus suppose $z_s = 10{,}000$ ft, $c_0 - \bar{c} = 18$ ft/sec, while $u_0 - u = 50$ ft/sec. Then $R = 47{,}000$ ft, while R' is infinite.

The extension of the above results to include examination of audibility at all points on the surface is not difficult. The details will be found in Milne (loc. cit., p. 108). In particular we may note that with the wind (as above) in the positive x direction, if R_φ denotes the range of audibility in the direction making an angle φ with the x direction, it develops that

$$R_\varphi = z_s \sqrt{\frac{\bar{c}}{c_0 - \bar{c} + u_0 - \bar{u} \cos \varphi}}.$$

A sketch of the curves obtained by plotting R_φ as a function of φ for

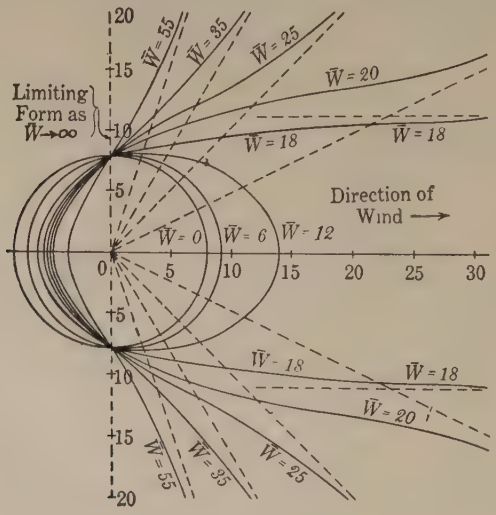

Fig. 12·8.

various values of $\bar{u}$ and using the above data for $c_0 - \bar{c}$ is given in Fig. 12·8. For $\bar{u} < c_0 - \bar{c} + u_0$, the curves are closed ovals. The

sound is audible at all points within but inaudible everywhere else. Note that the effect of the component of u perpendicular to the lines $\varphi =$ constant is neglected. For $\bar{u} > c_0 - \bar{c} + u_0$, the curves are unclosed but still divide the region into zones of audibility and inaudibility. Each curve has a pair of asymptotes given by

$$\varphi = \pm \text{ arc cos } \left[\frac{(c_0 - \bar{c} + u_0)}{\bar{u}} \right].$$

Other cases of limited audibility possessing mainly theoretical interest will be found in Milne, p. 109 ff.

12·4. Sound Ranging.—Practical Details.—Shell Sounds and Miscellaneous Phenomena.—The preceding section deals mainly with the theoretical corrections to the observed direction of an aerial sound source. Here we shall note a few practical details. During the late war much work was done on the location of cannon by the sounds received and the times of reception at three or more listening stations. Consulting Fig. 12·9, let S denote the source of the sound and let A, B, C be three receiving stations containing microphones [1] or other listening devices which are capable of fixing

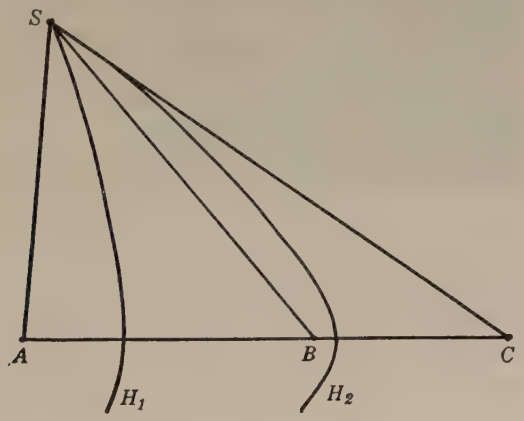

Fig. 12·9.

accurately the time of arrival of the sound from the gun. Now the locus of a point the difference of whose distances to two fixed points is constant is a hyperbola with the two points as foci. Hence the

[1] See, for example, Section 10·12.

source S must lie on the hyperbola with A and B as foci and with the characteristic difference $(t_B - t_A)c$, where t_A is the time of arrival at A and t_B that at B, and c is the velocity of sound. Similarly S must also lie on the corresponding hyperbola with B and C as loci. Hence it must lie at the point of intersection of the two hyperbolas, which fixes the position of the source.

As a matter of fact difficulties arise in the actual use of the above simple principle. When the sound source is directly ahead or nearly so, errors will occur which are similar to those which we have already mentioned in connection with subaqueous localization (Sec. 10·14, 10·15). Then there are the problems connected with the arrangement of an exact timing device. These will not be entered into here, but we may discuss at greater length the principal difficulty involved, because it introduces a type of sound transmission we have not previously mentioned in this chapter. (But see Sec. 1·20.) The sound wave received from the discharge of a gun is not a single signal but consists generally of *three* separate signals. The first is the *head* or *bow* wave, forming the envelope of waves emitted by a projectile as it progresses with a velocity greater than the normal velocity of sound. The following figure (Fig. 12·10)[1] illustrates this effect by means of a photograph of a moving bullet. The bow wave reaches the observer first as a sharp crack. The second is the wave due to the fall or explosion of the shell, while the third is the actual gun wave due to the expanding powder gases at the muzzle of the gun and traveling as far as the observer is concerned with the normal speed of sound. Since it is only the gun wave which gives the desired information about the location of the source, the fact that all three waves are recorded on the receiving device is a matter of embarrassment. However, it was found by Esclangon[2] and others that by using

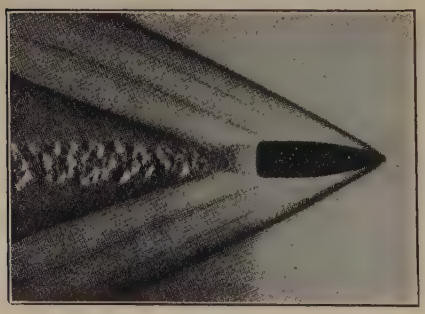

Fig. 12·10.

[1] Taken with kind permission from "Akustik," Vol. VIII of the "Handbuch der Physik," Verlag von Julius Springer, Berlin, 1927.

[2] E. Esclangon, Revue Scientifique, 115, 369, 1921.

a manometer resonator of large capacitance (i.e., large air volume) the gun wave could be separated from the others by the much larger response which it evokes. In this connection the use of Tucker's hot wire microphone [1] to detect the large air motions in the resonator has proved of great value. The corrections due to wind and temperature have already been sufficiently emphasized in Section 12·3.

The importance of the binaural phase effect in sound signalling and ranging has already been stressed in connection with subaqueous acoustics (Sec. 10·14). This method has proved of equal value for localization in air, particularly for that of airplanes. The customary set-up involves four long horns (conical or exponential), two arranged on a horizontal axis for an azimuth determination and two on a vertical axis for elevation determination. For night use the horns may be coupled by means of a unified control system to a searchlight battery and good accuracy in "spotting" has been reported.[2]

In connection with the character of aircraft sounds it may be appropriate here to recall the observations of Stewart,[3] who found that under poor listening (daytime) conditions such sounds are confined to the lower frequencies of the actual engine noises, while at night under good listening conditions the high frequencies are far more prominent than the low. In the former case the explanation is undoubtedly the greater scattering of the high frequency components by the irregularity of the atmosphere (Secs. 1·9 and 1·11). In the latter case, according to Stewart, the question is one of audition. Sabine [4] found that sounds of frequency 64 and 1024 appear to the ear to be equally loud if they are, respectively, 7×10^4 and 15×10^6 times their minimum audible intensity. Hence, since the sound intensity in a homogeneous atmosphere falls off inversely as the square of the distance from the source, the sound of lower frequency will be the first to fall below its minimum audible intensity and would cease to be heard before the sound of higher frequency.

The interesting phenomena connected with explosion waves have already been touched on briefly in Section 1·9 and numerous

[1] See Section 8·4.

[2] See the reported observations of G. W. Stewart, Phys. Rev., 14, 166, 1919.

[3] G. W. Stewart, Phys. Rev., 14, 376, 1919.

[4] W. C. Sabine, Contributions, Phys. Laboratory, Harvard Univ., No. 8, 1900.

references given. The principal anomalies connected with such waves are (*a*) the abnormally large sound velocity in the neighborhood of the source, (*b*) a second zone in which the velocity is normal save for wind and temperature effects, (*c*) a third zone of inaudibility, and (*d*) a fourth zone of renewed audibility. More complete details will be found in the literature.[1]

Questions and Problems

1. What factors are involved in the reflection of sound "from a cloud"? Discuss their probable magnitude. Under what circumstances will an aerial echo be produced?

2. Enumerate the reasons favoring the elevation of bells and chimes in towers.

3. Why would not a cloud have the same effect in preventing transmission as the same amount of water in the form of a film?

4. Why do the irregularities in the planity of the strata in the atmosphere scatter the short waves more than the long waves?

5. Does the refraction caused by wind and by temperature vary with the frequency employed and why?

6. Assume a source of sound next the earth on a plane surface. Discuss the influence of wind and temperature on the shape of the wave front.

[1] For example, E. G. Richardson, Sound, pp. 20–25. Also, Handbuch der Physik, Berlin, 1927, Vol. VIII. Akustik, pp. 677–680.

APPENDIX I

Table of Values of Acoustical Data

c = velocity of sound in meters/sec. at 20° C., where not otherwise stated.
ρ_0 = density in grams/cm³.

$R = \rho_0 c$, specific acoustic resistance in $\frac{\text{gram}}{\text{sec cm}^2}$.

The substances are arranged in each group in the order of decreasing R.

Solids	c	ρ_0	$R \times 10^{-4}$
Platinum	2690	21.4	575
Nickel	4973	8.6–8.9	427–443
Cobalt	4724	8.7	412
Gold (hard)	2100	19.3	406
Steel	5000	7.8	390
Glass (upper limit)	6000	5.9	354
Gold (soft)	1743	19.3	333
Copper	3560	8.9	317
Brass	3500	8.5	298
Silver	2610	10.5	273
Zinc	3700	7.1	262
Cadmium	2307	8.6	199
Tin	2500	7.3	183
Lead	1227	11.4	140
Aluminum	5104	2.6	132
Glass (lower limit)	5000	2.4	120
Marble	3810	2.6	99
Brick	3652	1.4–2.2	51–80

Wood varies with the kind, specimen and relation of the wave direction to the grain.

Solids	c	ρ_0	$R \times 10^{-4}$
Upper limit—Ash	4670	0.85	40
Lower limit—Elm	1013	0.54	5
Paraffin	1300	0.90	11
Tallow	390	0.95	3.7
Cork	500	0.24	1.2
Wax, 17° C.	880	0.96	0.84
Wax, 28° C.	440	0.96	0.44
* Rubber	31–69	0.95	0.29–0.66

* Variable dependent on color and manufacture, as well as on temperature.

Liquids	c	ρ_0	$R \times 10^{-4}$
Mercury	1407	13.6	191
Water (sea)	1490	1.025	15.3
Chloroform	1032	1.48	15
Water (air free)	1461	1.00	14.6
Alcohol, 12.5° C.	1241	0.81	10
Benzol, 17° C.	1166	0.90	10

Gases	c	ρ_0	$R \times 10^{-4}$
Chlorine, 0° C.	206	0.00317	0.0065
Carbon Dioxide, 0° C.	258	0.00198	0.0050
Oxygen, 0° C.	317	0.00143	0.0045
Air (dry, 20° C., 760 mm)	344	0.001205	0.0042
Carbon Monoxide, 0° C.	337	0.00125	0.0030
Methane, 0° C.	432	0.00072	0.0030
Steam, 100° C.	405	0.00058	0.0023
Hydrogen, 0° C.	1270	0.00009	0.0011

APPENDIX II

Propagation of Elastic Waves in an Isotropic Solid

The propagation of elastic waves through a solid medium is in general an extremely complicated problem. Fortunately in the case of an isotropic solid the matter is somewhat simplified, for here we need but two elastic constants (the others being expressible in terms of these), namely, the shear modulus n, defined as the ratio of shearing (i.e., tangential) stress to shearing strain (i.e., angle of shear), and the bulk modulus E, which has been defined in Section 1·13.

The remaining elastic constants of practical importance are Young's modulus Y and Poisson's ratio σ, which is defined [1] as the ratio of the lateral strain to the longitudinal strain. The fundamental relations connecting E, n, Y and σ can be shown to be

[1] See, for example, Love, Mathematical Theory of Elasticity, 3d edition, p. 101. Also the somewhat simpler treatment in Edser, General Physics for Students, p. 223.

$$n = Y/2(\sigma + 1), \tag{II·1}$$

$$E = Y/3(1 - 2\sigma), \tag{II·2}$$

whence

$$Y = 9En/(n + 3E), \tag{II·3}$$

$$\sigma = (3E - 2n)/2(n + 3E), \tag{II·4}$$

on reduction.

Let the component displacements of any point in the solid along the three coördinate axes be ξ, η, ζ, respectively. For the external stresses we shall denote by X_x the normal stress on the yz plane in the x direction, while the tangential stresses on this plane in the y and z directions are, respectively, Y_x and Z_x. The corresponding quantities for the xz plane are Y_y, X_y and Z_y; and for the xy plane Z_z, X_z, and Y_z, respectively. By considering the various stresses on the faces of a volume element $\Delta x \Delta y \Delta z$, we have for the total force (due to stress) acting on the element in the x, y, and z directions, respectively:

$$\begin{aligned}
&\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right)\Delta x \Delta y \Delta z,\\
&\left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right)\Delta x \Delta y \Delta z,\\
&\left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right)\Delta x \Delta y \Delta z.
\end{aligned} \tag{II·5}$$

Now let us consider the deformation of the parallelepiped element under the influence of the stress forces by the application of Hooke's law.

The elongations under the stresses X_x, Y_y, and Z_z, along the x, y, and z axes, respectively, are in accordance with this law:

$$\begin{aligned}
\frac{\partial \xi}{\partial x} &= X_x/Y - \sigma(Y_y + Z_z)/Y,\\
\frac{\partial \eta}{\partial y} &= Y_y/Y - \sigma(X_x + Z_z)/Y,\\
\frac{\partial \zeta}{\partial z} &= Z_z/Y - \sigma(Y_y + X_x)/Y.
\end{aligned} \tag{II·6}$$

Adding the above equations, utilizing thereafter the individual equations to solve for X_x, Y_y, and Z_z, respectively, in terms of the elongations, and making use of eqs. (II·1) and (II·2) to express all

constants in terms of n and E, we finally have the following expressions:

$$X_x = \left(E - \frac{2}{3}n\right)\delta + 2n\frac{\partial \xi}{\partial x},$$
$$Y_y = \left(E - \frac{2}{3}n\right)\delta + 2n\frac{\partial \eta}{\partial y}, \qquad \text{(II·7)}$$
$$Z_z = \left(E - \frac{2}{3}n\right)\delta + 2n\frac{\partial \zeta}{\partial z},$$

wherein we have set

$$\delta = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}, \qquad \text{(II·8)}$$

which is known as the "dilatation" of the element. The eqs. (II·7) take care of the elongation stresses and strains. The effect of the components X_y, Y_x, etc., is to produce shearing action. Applying Hooke's law again in the form that the shearing stress equals the shear modulus times the shearing strain, we have by examination of the elementary parallelepiped the following relations:[1]

$$X_y = Y_x = n\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right), \qquad X_z = Z_x = n\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right),$$
$$Y_z = Z_y = n\left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}\right). \qquad \text{(II·9)}$$

The quantities $(\partial\eta/\partial x + \partial\xi/\partial y)$, etc., are the shearing strains. The equality of X_y and Y_x, etc., is due to the assumption that the element is in equilibrium with respect to rotation. Introducing now the above values of X_x, X_y, etc., into the force expressions (II·5), we finally have

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = \left(E + \frac{n}{3}\right)\frac{\partial \delta}{\partial x} + n\nabla^2\xi,$$
$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} = \left(E + \frac{n}{3}\right)\frac{\partial \delta}{\partial y} + n\nabla^2\eta, \qquad \text{(II·10)}$$
$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = \left(E + \frac{n}{3}\right)\frac{\partial \delta}{\partial z} + n\nabla^2\zeta.$$

[1] L. Page, Introduction to Theoretical Physics, 1928, pp. 133, 134. Also Love, loc. cit., pp. 99, 100. Note the difference in notation.

The equations of motion of the solid due to the internal stresses are then

$$\rho\ddot{\xi} = \left(E + \frac{n}{3}\right)\frac{\partial\delta}{\partial x} + n\nabla^2\xi,$$

$$\rho\ddot{\eta} = \left(E + \frac{n}{3}\right)\frac{\partial\delta}{\partial y} + n\nabla^2\eta, \qquad \text{(II·11)}$$

$$\rho\ddot{\zeta} = \left(E + \frac{n}{3}\right)\frac{\partial\delta}{\partial z} + n\nabla^2\zeta.$$

Now if we differentiate the first of the above equations with respect to x, the second with respect to y, and the third with respect to z and add, we have

$$\rho\ddot{\delta} = \left(E + \frac{4}{3}n\right)\nabla^2\delta, \qquad \text{(II·12)}$$

since

$$\frac{\partial}{\partial x}\left(\nabla^2\xi - \frac{\partial\delta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nabla^2\eta - \frac{\partial\delta}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nabla^2\zeta - \frac{\partial\delta}{\partial z}\right) = 0.$$

But eq. (II·12) is the general wave equation already discussed in Section 1·12. Hence the dilatation δ is propagated in the form of a wave with velocity

$$c_L = \sqrt{\frac{E + \frac{4}{3}n}{\rho}}. \qquad \text{(II·13)}$$

That this corresponds to a longitudinal wave is easily seen by considering the special case in which there is no displacement in the y or z direction. The eqs. (II·11) then reduce to

$$\rho\ddot{\xi} = \left(E + \frac{4}{3}n\right)\frac{\partial^2\xi}{\partial x^2}, \qquad \text{(II·14)}$$

the usual equation for a longitudinal wave in the x direction.

In the case of a cylinder with its axis along the z direction, if we twist it in the xy plane, we get a wave of torsion traveling down the cylinder. For now

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y} = \frac{\partial\zeta}{\partial z} = 0,$$

and our equations become

$$\rho\ddot{\xi} = n\nabla^2\xi, \qquad \text{(II·15)}$$

$$\rho\ddot{\eta} = n\nabla^2\eta,$$

corresponding to a transverse wave traveling with velocity $c_T = \sqrt{n/\rho}$. It is clear that the torsional wave has a smaller velocity than a dilatational wave in the same substance.

The case of a long narrow rod has already been discussed in the text (Sec. 4·5). If it is subjected to longitudinal stress in the direction of its length, say the x axis, the deformations along the y and z directions will be negligible and we shall have simply in accordance with Hooke's law

$$X_x = Y \frac{\partial \xi}{\partial x}, \tag{II·16}$$

where Y is Young's modulus, while

$$Y_y = Z_z = 0.$$

The equation of motion then becomes

$$\rho \ddot{\xi} = Y \frac{\partial^2 \xi}{\partial x^2}, \tag{II·17}$$

and we have a longitudinal wave of velocity $c' = \sqrt{Y/\rho}$. Comparison shows that for a large number of solids c_L and c' differ by about 10% roughly, c_L of course being the larger. The specific acoustic resistance for a plane dilatational wave in a solid medium is thus in general given by

$$\rho_0 c_L = \sqrt{\left(E + \frac{4}{3}n\right)\rho_0}, \tag{II·18}$$

or in the particular case of solid rods more accurately by

$$R = \sqrt{E\rho_0}. \tag{II·19}$$

APPENDIX III

Mathematical Details in the Theory of Horns

The eq. (6·8) of Section 6·1, viz.,

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial(\log S)}{\partial x} \cdot \frac{\partial p}{\partial x} + k^2 p = 0, \tag{III·1}$$

is a linear differential equation of the second order. If we have two independent solutions of this equation in the form $u(kx)$ and $v(kx)$,

then from the theory of differential equations [1] the complete integral will be in the form

$$p = Au + Bv, \qquad \text{(III·2)}$$

where A and B are arbitrary.[2] Similarly since

$$\xi = \frac{1}{k^2c^2\rho_0} \cdot \frac{\partial p}{\partial x},$$

we have

$$\xi = \frac{A}{k^2c^2\rho_0}\frac{\partial u}{\partial x} + \frac{B}{k^2c^2\rho_0}\frac{\partial v}{\partial x}$$
$$= \frac{A}{\beta}u' + \frac{B}{\beta}v', \qquad \text{(III·3)}$$

where

$$\beta = kc^2\rho_0 \qquad \text{and} \qquad u' = \frac{\partial u}{\partial(kx)}, \qquad v' = \frac{\partial v}{\partial(kx)}.$$

The x coördinates of the two ends of the tube will be, respectively, x_1 and x_2. The end values of u and v are denoted by u_1 and u_2, v_1 and v_2, respectively. We now form the six determinants

$$D_1 = \begin{vmatrix} u_1 v_1 \\ u_1' v_1' \end{vmatrix}; \qquad D_2 = \begin{vmatrix} u_2 v_2 \\ u_2' v_2' \end{vmatrix}; \qquad D_3 = \begin{vmatrix} u_1 v_1 \\ u_2' v_2' \end{vmatrix},$$
$$D_4 = \begin{vmatrix} u_2 v_2 \\ u_1' v_1' \end{vmatrix}; \qquad D_5 = \begin{vmatrix} u_1 v_1 \\ u_2 v_2 \end{vmatrix}; \qquad D_6 = \begin{vmatrix} u_1' v_1' \\ u_2' v_2' \end{vmatrix},$$

which incidentally satisfy the relationship

$$D_1D_2 = D_1D_3 + D_5D_6. \qquad \text{(III·4)}$$

By means of these determinants it is possible to express A and B in terms of any two of the four quantities p_1, p_2, ξ_1 and ξ_2 (the subscripts referring to the two ends of the tube, as before). Thus we have

$$p_1 = Au_1 + Bv_1, \qquad p_2 = Au_2 + Bv_2,$$
$$\xi_1 = \frac{A}{\beta}u_1' + \frac{B}{\beta}v_1', \qquad \xi_2 = \frac{A}{\beta}u_2' + \frac{B}{\beta}v_2'. \qquad \text{(III·5)}$$

The plan is to solve the two equations at the left for A and B and

[1] See, for example, Whittaker and Watson, Modern Analysis, 3d ed., 1920, p. 194 f.

[2] The treatment here follows closely Webster, Proc. Nat. Acad. Sci., 5, 275, 1919.

then substitute into the equations at the right. We then obtain

$$p_2 = \frac{p_1 D_4 + \beta \xi_1 D_5}{D_1}, \tag{III·6}$$

$$\xi_2 = \frac{-p_1 D_6 + \beta \xi_1 D_3}{\beta D_1}. \tag{III·7}$$

We shall find it more convenient to write these in terms of the volume displacement $X = S\xi$. Then (III·6) and (III·7) may be written in the compact form

$$p_2 = ap_1 + bX_1, \tag{III·8}$$

$$X_2 = fp_1 + gX_1, \tag{III·9}$$

where

$$a = D_4/D_1, \qquad b = \frac{\beta}{S_1} \cdot \frac{D_5}{D_1},$$

$$f = -\frac{S_2}{\beta} \cdot \frac{D_6}{D_1}, \qquad g = \frac{S_2}{S_1} \cdot \frac{D_3}{D_1}.$$

Let the terminal impedances be Z_1 and Z_2. That is,

$$Z_1 = \frac{p_1}{X_1}, \qquad Z_2 = \frac{p_2}{X_2}. \tag{III·10}$$

Z_1 can be then expressed in terms of Z_2 or vice versa. This has been done in the text, Section 6·1, eqs. (6·13) and (6·14).

APPENDIX IV

Branch Transmission Theory of Acoustic Filtration [1]

If we consider the propagation of sound waves through a cylindrical tube as given by the fundamental eq. (1·17) of Sec. 1·13

$$\ddot{\varphi} = c^2 \frac{\partial^2 \varphi}{\partial x^2}, \tag{IV·1}$$

the complete solution for the velocity potential is then in the form

$$\varphi = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}. \tag{IV·2}$$

[1] This theory is based on that of W. P. Mason, Bell System Technical Journal, 6, 258, 1927, but is presented in a modified form.

The excess pressure (eq. (1·13), Sec. 1·12) becomes

$$\begin{aligned} p &= -\rho_0 \dot{\varphi} \\ &= -i\omega\rho_0[Ae^{-ikx} + Be^{ikx}]e^{i\omega t}, \end{aligned} \tag{IV·3}$$

while the volume current $\dot{X} = S\dot{\xi}$, where S is the cross-sectional area of the tube or conduit, is (eq. (1·8) of Sec. 1·12)

$$\begin{aligned} \dot{X} &= S\frac{\partial \varphi}{\partial x} \\ &= -ikS[Ae^{-ikx} - Be^{ikx}]e^{i\omega t}. \end{aligned} \tag{IV·4}$$

In order to evaluate the arbitrary constants A and B, let us call the pressure and volume current at $x = 0$, respectively, $p_1 e^{i\omega t}$ and $\dot{X}_1 e^{i\omega t}$. On substitution and reduction there results

$$A = i/2 \cdot (\dot{X}_1/kS + p_1/\rho_0\omega), \tag{IV·5}$$

$$B = -i/2 \cdot (\dot{X}_1/kS - p_1/\rho_0\omega). \tag{IV·6}$$

On resubstitution there develops

$$p = (p_1 \cos kx - iZ\dot{X}_1 \sin kx)e^{i\omega t}, \tag{IV·7}$$

$$\dot{X} = (\dot{X}_1 \cos kx - i/Z \cdot p_1 \sin kx)e^{i\omega t}, \tag{IV·8}$$

where we have set

$$\rho_0 c/S = Z, \tag{IV·9}$$

the acoustic resistance of the plane wave in the conduit.

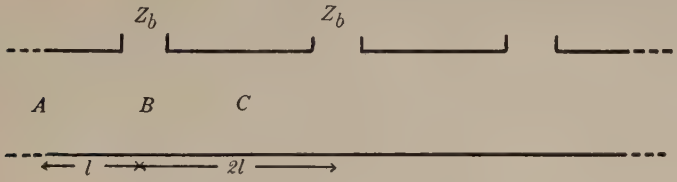

Fig. IV·1.

Consulting the figure (Fig. IV·1), which represents an acoustic conduit supposed to originate at A and extend to ∞, at distance l from A (i.e., at B) there is a branch of impedance Z_b. Further branches follow recurrently at $2l$, $4l$, etc. The midpoint between branches is denoted by C, and $AC = 2l$ is the length of one "section" of the line or semi-infinite filter. As in our discussion of

transmission through a conduit with an attached branch (Sec. 5·1) we assume the fundamental boundary conditions,

(1) Continuity of pressure at each junction.
(2) Continuity of volume current at each junction.

We now adopt the following notation. The pressure and volume current at A will be denoted by p_1 and $\dot{X}_1$ (the time factor $e^{i\omega t}$ is here left out); at B, the corresponding quantities at the junction in the conduit to the *left* will be p_{12} and $\dot{X}_{12}$, while at the junction in the conduit to the *right* they will be p_{21} and $\dot{X}_{21}$. Let the corresponding quantities at the junction in the *branch* be p_b and $\dot{X}_b$. Finally at C, the terminus of the first section and beginning of the second, let the pressure and volume current be p_2 and $\dot{X}_2$. The boundary conditions (1) and (2) then become

$$p_{12} = p_b = p_{21}, \tag{IV·10}$$

$$\dot{X}_{12} = \dot{X}_b + \dot{X}_{21}. \tag{IV·11}$$

But introducing the impedance $Z_b = p_b/\dot{X}_b$, we write

$$\dot{X}_{21} = \dot{X}_{12} - p_b/Z_b. \tag{IV·12}$$

Now from eqs. (IV·7) and (IV·8) we have (with the omission of the $e^{i\omega t}$ factor for the sake of simplicity)

$$p_{12} = p_1 \cos kl - iZ\dot{X}_1 \sin kl, \tag{IV·13}$$

$$\dot{X}_{12} = \dot{X}_1 \cos kl - i/Z \cdot p_1 \sin kl, \tag{IV·14}$$

whence by (IV·10) and (IV·12) it follows that

$$p_{21} = p_{12} = p_1 \cos kl - iZ\dot{X}_1 \sin kl, \tag{IV·15}$$

$$\dot{X}_2 = (\dot{X}_1 - p_1/Z_b) \cos kl - i(p_1/Z - Z/Z_b \cdot \dot{X}_1) \sin kl. \tag{IV·16}$$

Furthermore,

$$p_2 = p_{21} \cos kl - iZ\dot{X}_{21} \sin kl,$$

$$\dot{X}_2 = \dot{X}_{21} \cos kl - i/Z \cdot p_{21} \sin kl.$$

And on substitution these become after some reduction

$$p_2 = p_1(\cos 2kl + iZ/2Z_b \cdot \sin 2kl) - iZ\dot{X}_1(\sin 2kl + iZ/Z_b \cdot \sin^2 kl), \tag{IV·17}$$

$$\dot{X}_2 = \dot{X}_1(\cos 2kl + iZ/2Z_b \cdot \sin 2kl) - ip_1/Z \cdot (\sin 2kl - iZ/Z_b \cdot \cos^2 kl). \tag{IV·18}$$

Now since the filter is infinite in the direction to the right, we have

$$p_1/\dot{X}_1 = p_2/\dot{X}_2 = \cdots = Z_0, \qquad \text{(IV·19)}$$

where Z_0 is a constant impedance. That is, to the right of any mid-point such as C, the rest of the filter is precisely the same as it is to the right of any other mid-point. The expression for Z_0, which will be termed the *characteristic* impedance of the filter, is obtained as follows:

$$p_2/\dot{X}_2 = \frac{p_1 - iZ\dot{X}_1 \cdot \left(\dfrac{\sin 2kl + iZ/Z_b \cdot \sin^2 kl}{\cos 2kl + iZ/2Z_b \cdot \sin 2kl}\right)}{\dot{X}_1 - ip_1/Z \cdot \left(\dfrac{\sin 2kl - iZ/Z_b \cdot \cos^2 kl}{\cos 2kl + iZ/2Z_b \cdot \sin 2kl}\right)} = p_1/\dot{X}_1. \qquad \text{(IV·20)}$$

On solving (IV·20) for $p_1/\dot{X}_1 = Z_0$, we obtain finally

$$p_1/\dot{X}_1 = Z_0 = Z\sqrt{\frac{1 + iZ/2Z_b \cdot \tan kl}{1 - iZ/2Z_b \cdot \cot kl}}. \qquad \text{(IV·21)}$$

We now set

$$\cos W = \cos 2kl + iZ/2Z_b \cdot \sin 2kl, \qquad \text{(IV·22}a\text{)}$$

whence

$$\begin{aligned} \sin W &= \sin 2kl\sqrt{(1 - iZ/2Z_b \cdot \cot kl)(1 + iZ/2Z_b \cdot \tan kl)} \\ &= Z_0/Z \cdot (\sin 2kl - iZ/Z_b \cdot \cos^2 kl). \end{aligned} \qquad \text{(IV·22}b\text{)}$$

We can then write

$$p_2 = p_1 \cos W - i\dot{X}_1 Z_0 \sin W, \qquad \text{(IV·23)}$$

$$\dot{X}_2 = \dot{X}_1 \cos W - ip_1/Z_0 \cdot \sin W. \qquad \text{(IV·24)}$$

Since the symbols are arbitrary, we can write similar relations for any two adjacent sections. Thus,

$$p_{m+1} = p_m \cos W - i\dot{X}_m Z_0 \sin W, \qquad \text{(IV·25)}$$

$$\dot{X}_{m+1} = \dot{X}_m \cos W - ip_m/Z_0 \cdot \sin W. \qquad \text{(IV·26)}$$

Incidentally we can also establish by mathematical induction the following: [1]

$$p_{m+1} = p_1 \cos mW - i\dot{X}_1 Z_0 \sin mW, \qquad \text{(IV·27)}$$

$$\dot{X}_{m+1} = \dot{X}_1 \cos mW - ip_1/Z_0 \cdot \sin mW. \qquad \text{(IV·28)}$$

[1] Thus we prove by direct substitution that (IV·27) and (IV·28) are true for the special case where $m = 2$. Then assuming their truth for $m = n$, we can prove that they hold for $m = n + 1$. Hence they are established in general.

Going back to eqs. (IV·25) and (IV·26), we may write them in the form

$$p_{m+1} = p_m(e^{iW} + e^{-iW})/2 - \dot{X}_1 Z_0(e^{iW} - e^{-iW})/2,$$
$$\dot{X}_{m+1} = \dot{X}_m(e^{iW} + e^{-iW})/2 - p_1/Z_0 \cdot (e^{iW} - e^{-iW})/2,$$

which become, since $p_m/\dot{X}_m = Z_0$,

$$p_{m+1} = p_m e^{-iW}, \qquad \text{(IV·29)}$$
$$\dot{X}_{m+1} = \dot{X}_m e^{-iW}. \qquad \text{(IV·30)}$$

Now if W is real, the only change in pressure and volume current in going from section to section is one of phase, i.e., there is no attenuation. On the other hand, if W is complex, say $W = a + ib$, there will be attenuation along the filter. Let us suppose that Z_b is a pure reactance (i.e., no resistance or dissipation in the branch). Then $iZ/2Z_b$ will be real and consequently (IV·22*a*) cos W is always real. If $W = a + ib$, we have

$$\cos W = \cos a \cosh b - i \sin a \sinh b. \qquad \text{(IV·31)}$$

If $a = n\pi$ (n any integer or zero), the attenuation is thus given by

$$\cosh b = \pm (\cos 2kl + iZ/2Z_b \cdot \sin 2kl), \qquad \text{(IV·32)}$$

while if there is no attenuation, i.e., $b = 0$, the phase change is given by

$$\cos a = \cos 2kl + iZ/2Z_b \cdot \sin 2kl. \qquad \text{(IV·33)}$$

Since the transmission is without attenuation for W real, it follows that there is no attenuation for values of W such that

$$-1 < \cos W < +1, \qquad \text{(IV·34)}$$

and the limits of the non-attenuation region will then be given by

$$\cos 2kl + iZ/2Z_b \cdot \sin 2kl = \pm 1. \qquad \text{(IV·35)}$$

Another way of looking at the condition for transmission is to be found from a consideration of the characteristic impedance Z_0. Thus examining the expression for sin W (eq. IV·22*b*), we see that, if W is real, sin W must be real and consequently Z_0 must be real. The condition for transmission without attenuation is therefore that the characteristic impedance Z_0 shall be *real*.

APPENDIX V

Graphical Method for Obtaining Transmission Bands in Acoustic Filters

In the filter theory presented in the Sections 7·2 to 7·5, the region of no attenuation is given by the limiting values

$$Z_1/Z_2 = 0 \qquad \text{and} \qquad Z_1/Z_2 = -4.$$

We may simplify the discussion materially by replacing the above by the statement that for a non-attenuation region the ratio Z_1/Z_2 must satisfy the inequality relation

$$-1 \leq \frac{Z_1}{4Z_2} \leq 0. \tag{V·1}$$

In words, the transmission band must include those frequencies for which the line and branch impedances are of opposite sign and the absolute value of the line impedance is not greater than four times the absolute value of the branch impedance.

We now seek to prove a general theorem concerning the impedance of networks,[1] namely, that the impedance of any non-dissipative impedance (i.e., reactance) network always has a positive slope with the frequency as well as discontinuous changes from positive to negative infinity at anti-resonant frequencies (i.e., those for which $Z = \infty$), and may always be represented either by a number of simple (series M and C) resonant components in parallel or simple (parallel M and C) anti-resonant components in series. (Note that when M and C are in parallel, we have seen that $Z = i\omega M/(1 - Mc\omega^2)$. Such an impedance is what is meant here by an anti-resonant component.) Let Z' and Z'' be two impedances (of the kind mentioned) and let Z_s be their joint impedance in series and Z_p their joint impedance in parallel. Then

$$\left.\begin{aligned} Z_s &= Z' + Z'', \\ \text{while} \qquad Z_p &= \frac{Z'Z''}{Z' + Z''}. \end{aligned}\right\} \tag{V·2}$$

Differentiating with respect to frequency ν, we have

[1] Zobel, Bell System Tech. Jour., 2, 1, 1923.

$$\frac{dZ_s}{d\nu} = \frac{dZ'}{d\nu} + \frac{dZ''}{d\nu}, \qquad (V\cdot3)$$

$$\frac{dZ_p}{d\nu} = \frac{Z''^2}{(Z' + Z'')^2}\frac{dZ'}{d\nu} + \frac{Z'^2}{(Z' + Z'')^2}\frac{dZ''}{d\nu}. \qquad (V\cdot4)$$

Now since Z' and Z'' are both reactances, they will be of either the form $i\omega M$ or $-i/\omega C$. Now $d(i\omega M)/d\nu = 2\pi i M$ and is positive, and $d(-i/\omega C)/d\nu = i/2\pi C\nu^2$ and is also positive. Therefore, both $dZ'/d\nu$ and $dZ''/d\nu$ are positive, whence by (V·3) and (V·4) $dZ_s/d\nu$ and $dZ_p/d\nu$ must also be positive and Z_s and Z_p have positive slopes with respect to frequency. But any combination of reactances can be produced by additions in parallel or in series. Hence the theorem is proved.

With the use of this theorem it is possible to determine the nature of a given filter, i.e., to ascertain whether it belongs to the low pass, high pass or single band type. As a first illustration consider the simple ideal case represented in the accompanying diagram (Fig. V·1). The line impedance is here a simple inertance, i.e., Z_1

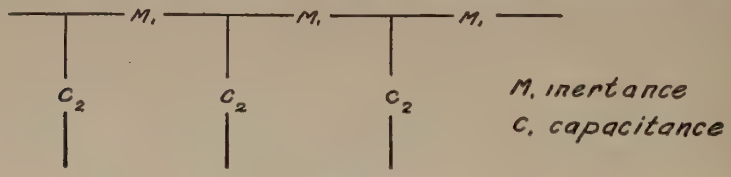

Fig. V·1.

$= i\omega M_1$, while the branch impedance is a simple capacitance, i.e., $Z_2 = -i/\omega C_2$. It is hardly necessary to stress that this is a purely ideal illustration. In Fig. V·2, Z/i is plotted against ω both for Z_1

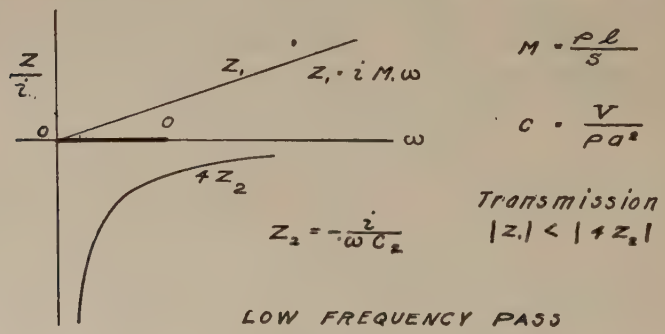

Fig. V·2.

and $4Z_2$. From the figure it is evident that the region of no attenuation extends from O to O', where at O' the ordinate of Z_1 is equal and opposite to that of $4Z_1$. For it is only in this interval that we have Z_1 and Z_2 of opposite sign and at the same time $|Z_1| \leqq 4|Z_2|$. The limits of the non-attenuation region are then $\nu_1 = 0$ and $\nu_2 = (1/\pi)\sqrt{1/M_1C_2}$ (obtained by putting $|Z_1| = 4|Z_2|$). It is interesting to note that the limiting frequency ν_2 is just twice the resonant frequency of M_1 and C_2 in series. It is thus evident that the filtering action is not strictly a resonance phenomenon, although it uses the elements of resonance. The ideal filter here pictured is thus a low-frequency pass type.

As a more practical example, let us note that represented in Fig. 7·6 of Section 7·3. Here we have

$$Z_1 = i\omega M_1 \qquad \text{and} \qquad Z_2 = i\left(\omega M_2 - \frac{1}{\omega C_2}\right).$$

Plotting Z_1/i and $4Z_2/i$ as usual, we obtain Fig. V·3. As before it is

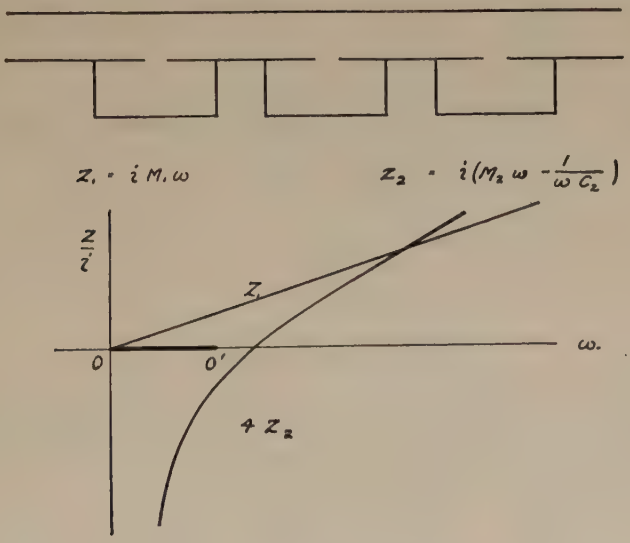

Fig. V·3. Single Band Filter

seen that the filter is of the low-frequency pass type. The limits of the non-attenuation region are $\nu_1 = 0$ and

$$\nu_2 = \frac{1}{\pi}\sqrt{\frac{1}{C_2(M_1 + 4M_2)}}.$$

The corresponding electrical line may be represented as in the following diagram (Fig. V·4), where ~~~~ signifies as usual

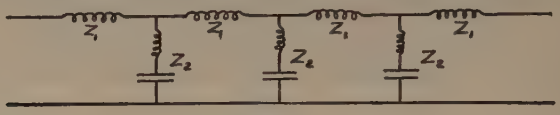

Fig. V·4.

inductance (or inertance in the acoustical case) and ⊣⊢ signifies capacitance.

Next let us apply the graphical method to the filter discussed in Section 7·4 and graphically represented in Fig. V·5. Here we set

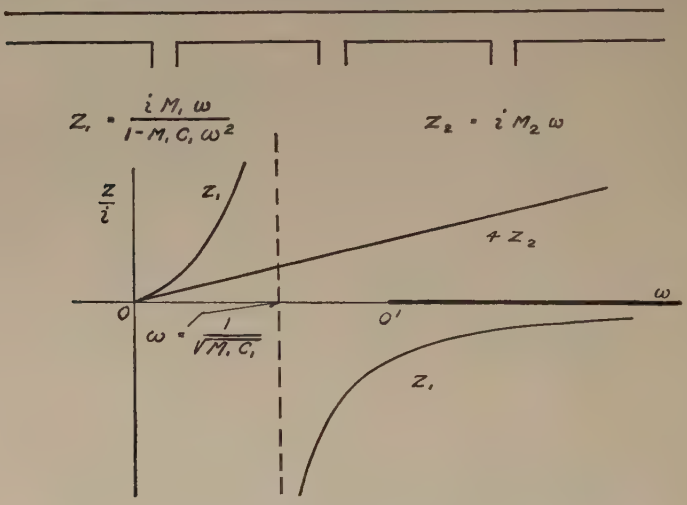

Fig. V·5. High Frequency Pass

$Z_1 = iM_1\omega/(1 - M_1C_1\omega^2)$ and $Z_2 = i\omega M_2$. We plot as usual Z_1/i and $4Z_2/i$. The first yields a curve which has an infinite discontinuity at $\omega = 1/\sqrt{M_1C_1}$ and thereafter has Z_1/i negative with steadily decreasing absolute value. In this case, $4Z_2$ is represented by a straight line. After O' is reached, $|Z_1|$ remains less than $|4Z_2|$ for all larger values of ω and so the transmission band extends from O' to ∞. That is, the filter is of the high-frequency pass type with the lower limit at[1]

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{4C_1M_2} + \frac{1}{M_1C_1}}.$$

[1] The limitations of the discussion in Section 7·4 must, of course, be kept in mind.

Finally, let us consider the type presented in Section 7·5 and depicted in Fig. V·6. Here $Z_1 = i\omega M_1$ and

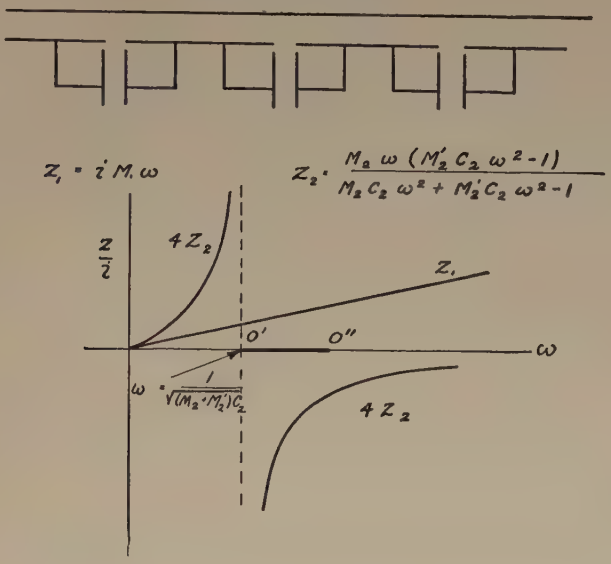

Fig. V·6. Low Frequency Pass

$$Z_2 = \frac{i\omega M_2(M_2'C_2\omega^2 - 1)}{M_2C_2\omega^2 + M_2'C_2\omega^2 - 1}.$$

The curve for $4Z_2$ shows an infinite discontinuity at

$$\omega_1 = \frac{1}{\sqrt{C_2(M_2 + M_2')}}.$$

The transmission band is clearly $O'O''$ with limits

$$\nu_1 = \frac{1}{2\pi}\sqrt{\frac{1}{C_2(M_2 + M_2')}}$$

and

$$\nu_2 = \frac{1}{2\pi}\sqrt{\frac{M_1 + 4M_2}{C_2(M_1M_2 + M_1M_2' + 4M_2M_2')}}.$$

It is of interest to note that the lower limit of the band is the anti-resonant frequency of the side branch.

APPENDIX VI

The Asymmetrical Vibrator

In the discussion of the mechanical oscillations of a membrane in Section 2·2 we assumed that the elastic restoring force is proportional to the first power of the displacement. Strictly speaking, the restoring force should be regarded as a power series expansion in the displacement, thus

$$F = f\xi + g\xi^2 + h\xi^3 + \cdots, \qquad \text{(VI·1)}$$

where under certain conditions g and h may have perceptible values. The inclusion of the term $g\xi^2$ implies that the vibrator is asymmetrical, the restoring force being different in magnitude for positive and negative displacements. According to Helmholtz the structure of the ear drum is of just such an asymmetrical character. We therefore should consider briefly the nature of the forced oscillations of such a system.

Suppose that two harmonic forces of frequencies $\nu_1 = \omega_1/2\pi$ and $\nu_2 = \omega_2/2\pi$ are impressed on the system. The equation of motion then becomes, neglecting damping,

$$m\ddot{\xi} + f\xi + g\xi^2 = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t. \qquad \text{(VI·2)}$$

Now the first approximation to the solution for the steady state is (Sec. 2·2)

$$\xi = \frac{F_1}{f - m\omega_1^2} \cos \omega_1 t + \frac{F_2}{f - m\omega_2^2} \cos \omega_2 t. \qquad \text{(VI·3)}$$

To get the second approximation (i.e., the effect of the term $g\xi^2$), we substitute from (VI·3) into this term and write the resulting equation

$$\begin{aligned} m\ddot{\xi} + f\xi = {} & F_1 \cos \omega_1 t + F_2 \cos \omega_2 t \\ & - g\left\{ \frac{F_1^2}{(f - m\omega_1^2)^2} \cos^2 \omega_1 t + \frac{F_2^2}{(f - m\omega_2^2)^2} \cos^2 \omega_2 t \right. \\ & \left. + \frac{2F_1F_2}{(f - m\omega_1^2)(f - m\omega_2^2)} \cos \omega_1 t \cos \omega_2 t \right\}. \end{aligned} \qquad \text{(VI·4)}$$

Letting $F_1/(f - m\omega_1^2) = f_1$ and $F_2/(f - m\omega_2^2) = f_2$ and making some trigonometrical reductions, we have for (VI·4)

$$m\ddot{\xi} + f\xi = F_1 \cos \omega_1 t + F_2 \cos \omega_2 t$$
$$- \frac{g}{2}(f_1^2 + f_2^2) - \frac{g}{2} f_1^2 \cos 2\omega_1 t - \frac{g}{2} f_2^2 \cos 2\omega_2 t \quad \text{(VI·5)}$$
$$- \frac{g f_1 f_2}{2}[\cos (\omega_1 + \omega_2)t + \cos (\omega_1 - \omega_2)t].$$

It is at once evident that the solution to eq. (VI·5) will contain in addition to the harmonic terms in ω_1 and ω_2 as given by the first approximation other harmonic terms of frequencies $2\omega_1$, $2\omega_2$ (i.e., octaves) and $(\omega_1 + \omega_2)$ and $(\omega_1 - \omega_2)$. It is the latter which are of significance in the theory of summation and difference tones in the Helmholtz theory of audition (see Sec. 9·8).

APPENDIX VII

Table of Absorption Coefficients (see Sec. 11·5)

This table is a contribution from Dr. P. E. Sabine and the Riverbank Laboratories and is presented here with their kind permission.

Material	Pitch					
	128	256	512	1024	2048	4096
	Coefficient					
1. Acousti-Celotex, Type *A* perforated fiber board, 13/16″ thick, 441 holes per sq. ft., 3/16″ diameter, 1/2″ deep, plain side exposed		.20	.21	.19	.17	.24
2. Acousti-Celotex, Type *B* same as Type *A*, but with perforations exposed		.27	.40	.46	.42	.42
3. Acousti-Celotex, Type *BB*, 1 15/16″ thick, 441 holes per sq. ft., 3/16″ diameter, 1 1/8″ deep		.39	.63	.73	.55	.46
4. Acousti-Celotex (tested 1924), 1″ thick, 400 holes per sq. ft., 3/4″ deep		.32	.46	.66	.53	.47
5. Akoustolith Tile, 7/8″ thick, fine texture, cemented to clay tile	.06	.22	.28	.48	.50	.31
6. Balsam Wool, soft wood fiber, paper backing, scrim facing, 1″ thick, .254 pounds/sq. ft	.10	.27	.50	.68	.56	.48
7. Same covered with perforated metal cover, 1/16″ holes, 64 per sq. in.	.09	.25	.48	.66	.57	.47
8. Standard Celotex, 7/16″ thick on 1″ furring		.16	.22	.20	.16	.15
9. The same on 2 × 4″ studs	.19	.14	.13	.14	.14	.16
10. Draperies, hung straight, in contact with wall, cotton fabric, 10 oz. per sq. yd	.03	.04	.11	.17	.24	.35
11. The same, cotton fabric, 14 oz. per sq. yd	.04	.07	.13	.22	.32	.35
12. The same, velour, 18 oz. per sq. yd	.05	.12	.35	.45	.38	.36
13. The same as No. 12, hung 4″ from wall	.06	.27	.44	.50	.40	.35
14. The same as No. 12, hung 8″ from wall	.08	.29	.44	.50	.40	.35

TABLE OF ABSORPTION COEFFICIENTS—(*Continued*)

Material	Pitch					
	128	256	512	1024	2048	4096
	Coefficient					
15. Cotton fabric, 14 oz./sq. yd., draped to 7/8 its area	.03	.12	.15	.27	.37	.42
16. The same as No. 15, draped to 3/4 area	.04	.23	.40	.57	.53	.40
17. The same as No. 15, draped to 1/2 area	.07	.31	.49	.81	.66	.54
18. Felt, Standard 1", all hair	.09	.34	.55	.66	.52	.39
19. Felt, Asbestos-Akoustikos (hair and asbestos fiber), 1/2" thick	.07	.14	.31	.51	.51	.43
20. The same 3/4" thick	.08	.23	.45	.65	.56	.46
21. The same 1" thick	.11	.31	.59	.68	.58	.46
22. The same 1 1/2" thick	.13	.41	.73	.73	.58	.46
23. The same 2" thick	.21	.46	.79	.75	.58	.46
24. The same 3" thick	.33	.56	.79	.77	.58	.46
25. Flax-linum, semi-stiff flax fiber board, 1/2" thick	.09	.15	.34	.57	.51	.47
26. Masonite, Standard 1/2" board (Pressed wood fiber), laid on 1" furring, 18" O. C.	.09	.30	.33	.32	.30	.37
27. The same nailed to 2 × 4", 16" O. C.	.16	.26	.34	.36	.30	.25
28. The same nailed to 1 × 2" furring, 16" O. C.	.15	.26	.31	.32	.30	.28
29. 1" Nashkote *AAX*, 1" felt with cotton fabric cemented on surface, two coats, special paint	.11	.25	.34	.46	.48	.36
30. Nashkote *B*-332, 1" felt with perforated oil-cloth, 3/32" perforations, 10 per sq. in.	.11	.31	.67	.81	.64	.50
31. Plaster, gypsum on wood lath on wood studs, rough finish	.016	.032	.039	.050	.030	.028
32. The same, with smooth finish ("lime putty")	.020	.022	.032	.039	.039	.028
33. Plaster, lime on wood lath on wood studs, rough finish	.027	.046	.060	.085	.043	.056
34. The same, smooth finish	.024	.027	.030	.037	.019	.034
35. Plaster, "Calacoustic," 1/2" thick	.06	.10	.14	.15	.15	.20
36. Plaster, Sabinite, 1/2" thick	.06	.16	.21	.29	.34	.37
37. Stockade slab, wood fiber, cemented with magnesite, 1" thick	.11	.13	.27	.50	.63	.40
38. The same, 2" thick	.09	.35	.60	.67	.45	.53

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